FEDERAL UNIVERSITY OF TECHNOLOGY MINNA

DRIVING A CAR WITHOUT A DASHBOARD, YOUR OPINION IS YOURS!

By

PROFESSOR YUSUPH AMUDA YAHAYA

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Professor of Mathematics

INAUGURAL LECTURE SERIES 81

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COURTESIES, PROTOCOL AND PREAMBLE

OPENING QUOTE:

> ***"Life is movement; move forward with the current of your soul, be not discouraged, for you shall pass this way but once. Whatever befalls you, just don't stop; keep on moving."***
>
> \- Author Unknown.

The Vice-Chancellor, Deputy Vice-Chancellors, Registrar and other Principal Officers of the University, Deans of Postgraduate School and Student Affairs, Dean of School of Physical Sciences (SPS) and other Deans of Schools, Directors of Units and Centres, Professors and other members of Senate, Head of Department (HOD) of Mathematics and other HODs, Other members of academic staff, members of administrative and technical staff, my lords spiritual and temporal, members of my family - nuclear and extended, distinguished invited guests, gentlemen of the press, Great FUTMinna Students, ladies and gentlemen.

1.0 INTRODUCTION

This is the fourth inaugural lecture from the Department of Mathematics. Inaugural lectures bring the town and the gown together. The university gives a lecturer an opportunity to tell his "own Story" his background and his growing years up to the unique world of research, his discoveries and how he has succeeded in moving the local, national and or global academic community forward by at least an inch, sometimes a mile. The lecture usually ends with recommendations for University, Government and the larger society. Indeed, all those who have helped the "Don" in reaching his heights are acknowledged and appreciated.

In the university tradition the world over, inaugural lectures are delivered soon after lecturers have been admitted into the exclusive guild of "Professors." My Professorship was approved in 2014 but backdated to October 2013. Thus, I should have given this lecture some years ago, but I could not. Several factors were responsible for my inability to deliver this lecture before now. I would not want to bore you with these factors. Any such attempt would amount to rationalization. To me, it is better late than never. In any case, God's time is the best. This is the 81st in the series of inaugural lectures delivered by professors of this University. I acknowledge and appreciate the efforts of earlier lecturers. Some of the earlier inaugural lecturers focused on such areas as "Structural Engineering Practice and its Implication on Structural Integrity;" some had romanced with the solid earth by "subduing the earth with Exploration Geophysics;" one lecturer took us through the partnerships for food and nutritional security of the poor, the role of legumes and Rhizobia. One of them even warned us that, excess calorie intake or too little intake is very detrimental to our health; hence we must watch what we eat. Some focused on the celestial bodies and the arrangement that makes them rotate without crossing each other's path, in delicate balance and ordered by almighty Allah. Early 2019, a geographer took us on the climate change, paradoxes and mapping the -tragedy of the common. Since the position our society in general has taken with respect to "Mathematics" is well documented, I would try to describe some mathematical procedures as simply as possible but without giving turbulent derivations. This is to ensure that both town and gown are brought closely together here today.

Mr. Vice-Chancellor Sir, permit me to inform you and this gathering that my being a "Mathematician" is by accident. After my secondary education, I could not buy a "JAMB" form, thus I started work as "enumerator" on the government policy of property enumeration by the then Military Government of Gen. Ibrahim Babangida (rtd). After a year on the job, I borrowed money from Mr. Aliyu Yahaya (now called Mr. Sulaiman Dagbo who was working with the Kwara State College of Education, Ilorin). It may interest this august gathering that I finished paying the loan I used to buy JAMB form at the end of my 200 level in the university. Like every Ilorin-child, the course in vogue was either "LAW or MEDICINE", rush for Civil Engineering was just melting down! I applied to University of Ilorin to read Medicine and Surgery (MBBS) but was not offered admission, but with kind heart of my former Economics teacher at secondary school who had just moved to the Registry Department of University of Ilorin - **Mr. R. O. Olajide**. During my visit to him at his new position, he said "**Yahaya** your mate would finish second year of their programme if you lose this year again!" He took me to his friend - HOD Mathematics (**Dr. M. A. Ibiejugba**) who later became the first Professor of Mathematics at the University of Ilorin and now late (may his soul rest in peace). Despite my UTME subject combination of "**Biology, Chemistry and Physics**", the Department admitted me for B.Sc. single honours in Mathematics. Today, I give glory to God almighty as I stand before this gathering as a ***"Professor of Mathematics"***.

Mr Vice-Chancellor Sir, this lecture mainly focuses on two numerical integration techniques for solution of differential equation be it partial or ordinary (DEs). These are the Runge-Kutta Methods (RKM) and Linear Multistep Methods (LMM). While the RKM was originally derived as discrete schemes, same way the LMM originated as discrete schemes from operators. Both methods have been fussed into one, as they can now both be formulated as continuous through Multistep Collocation (MC). My research on the two integration techniques can be further classified into:

1. Linear Multistep Methods, including the combination of two methods, as predictor-corrector pairs.
2. The theory interrelating stability, consistency and convergence.

3. Runge-Kutta methods, detailed analysis of the order stability and convergences and reformulation into continuous form and vice-versa.
4. Hybrid Methods; a class whose computational potentialities have probably not yet been fully exploited.
5. Block formulation of Linear Multistep Methods (LMM) and its Hybrid form.

The classes of problem under consideration are the first order and special second order ordinary differential equations. The latter frequently occurs in the study of solar system, molecular biology, etc. Yahaya (2004) presented two categories of numerical integrators for this class of problem- namely Block Linear Multistep Methods (BLMM) and Block Hybrid Methods both arising from Single Continuous Linear Multistep Methods (CLMM).

These block formulations possess the desirable feature of Runge-Kutta Method (RKM) of being self-starting and less expensive in terms of number of functional evaluations in contrast to the conventional Runge-Kutta Methods. By their block nature the methods produce simultaneously, approximations to the ordinary differential equations either initial value or boundary value problems at both grid and off grid point where necessary.

Yahaya and Badmus (2009) presented some self-starting hybrid block methods with high orders for solution of general second and third order initial value problems. The continuous formulations enable the integrators to be further differentiated and evaluated at some other points. I want to start this lecture by providing some results from researches in which I was involved. Mr Vice-chancellor, Sir, I thus seek your indulgence to depart slightly from the convention of inaugural lectures in this University. The experiences which I want to share with you today derived from various sources: my experience as a school teacher at Okanle Fajeromi-a Secondary School in Kwara State; my experience as a Mathematics teacher at Kaduna Polytechnics; my experience as a University teacher; my experience as a visiting lecturer to institutions like Nigerian Defence Academy (NDA), Kaduna; Kano State University, Wudil, and some others.

Apart from my academic research activities, I have also supervised five (5) PhDs, co-supervised 3 PhDs, some (15) MSc/M.Tech and about (40) forty undergraduate projects. Some of them relate tangentially to my “talk" today. In any case, if you could find all of what I intend to say here today in a book or journal then my standing before you this afternoon would have been a waste of time.

Mathematics as a subject, call it science or art may be said to be “useful”. Its development affects, even indirectly, the social, psychological and material well-being of men; it promotes their comfort, happiness and help them to attain their goals in life, easily. Many young boys and girls have their dreams broken, because they could not make credit in Mathematics in school certificate examinations. Everybody here will agree with me that Medicine and Psychology are useful because they relieve suffering; structure/civil engineering is useful because it helps us to build houses and bridges and so raise the standards of life or destroy life. Telecommunication Engineering enables us to communicate through mobile communications, thus saves life by avoiding unnecessary movement from place to place. Mobile telephony is connecting the world and empowering the people, mobile broadband promises to be a greater driver of economy globally. Broadband refers to the bandwidth for transmission or reception of data, measured in the number of bit transmitted per second (bps).

Ladies and Gentlemen, ask people on the road to mention useful sciences, hardly will anyone mention Mathematics! But Mathematics is certainly useful because the engineers cannot do their job without a fair working knowledge of Mathematics. Evidently, Mathematics is the study of numbers, quantity, space structure and change. It is used as an essential tool in many fields including natural sciences, engineering, medicine and the social sciences. Applied Mathematics which is concerned with application of knowledge of mathematics to other fields, inspires and makes use of new mathematical discoveries and sometimes leads to the development of entirely new mathematical disciplines such as statistics and game theory. Mathematicians also engage in pure mathematics or mathematics for its own sake, without having any application in mind. There is no clear line of separating pure and applied mathematics, and practical applications for what began as pure mathematics may require some computations. Thus, for example, numerical linear algebra has evolved from linear algebra. Numerical analysis can also be defined as the study of algorithms that use numerical approximations for the problem of mathematical analysis. It is concerned with obtaining approximate solutions while maintaining reasonable bounds on error. Numerical analysis naturally finds application in all fields of engineering and physical sciences, but in the 21st century, the life sciences and even humanities have adopted elements of scientific computations. This is an issue that bothered me. I have decided, therefore, for lack of better title to use as my pivot, the topic. **"Driving a Car without a Dashboard, Your opinion is Yours!"** It is indeed, an important issue that requires constant re-examination and new approaches, for example in the early seventies (1970s) when oil engineers were digging “drywells" and the profit expected from an “oil well" was not being realized because a chemist will only carry out as at then “hydrocarbon" test to locate oil position underneath the earth, but after drilling it was observed that “Gaseous" hydrocarbon and the liquid hydrocarbon exhibited the same property. American drillers, then, employed the use of finite differences on the seismic data from that area to further ascertain presence of liquid hydrocarbon as against “the Gaseous” and thus put to rest the issue of “dry wells.” Today , report on petroleum potentiality is now available , further investigations by geophysicist on location of liquid hydrocarbon gave conditions that generally favour petroleum formation as, the existence of a sedimentary basin filled with thick sequences of sedimentary rocks. Abundant organic Matter-type within the fine-grained source rock that is capable of generating the hydrocarbons which must have been rapidly deposited in an anoxic marine environment.

2.0 TERMINOLOGIES IN COMPUTATIONAL MATHEMATICS AS CONTAINED IN NUMERICAL ANALYSIS

Mathematical modelling and simulation techniques are the trend world over. Today there are three major categories of scientists within each science discipline; namely experimental scientists, theoretical scientists and computational scientists. In mathematics we now have pure mathematicians, applied mathematicians and computational mathematicians. It is widely reported that multidisciplinary approaches to problems have contributed to the emergency of modern computational science. While in engineering-technology we have new emergence like Mechatronics from such background of Electrical and Mechanical Engineering. However, before proceeding to the main lecture of the day, let me briefly say something about the *'Catalyst'* that makes the journey very attractive.

2.1 Power Series

The functional series of the form

$$\sum_{k=0}^{+\infty} C_k x^k = C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n + \ldots \qquad 2.1.1$$

or of the form

$$\sum_{k=0}^{+\infty} C_k (x-x_0)^k = C_0 + C_1(x-x_0) + C_2(x-x_0)^2 + \ldots + C_n(x-x_0)^n + \ldots \quad 2.1.2$$

where the coefficient $C_0, C_1, \ldots, C_n, \ldots$ are constant numbers while x is a variable is called a *power series*, and it may converge,

(i) Only for the single value, $x = x_0,$

(ii) Absolutely for $|x-x_0| < \in$ i.e. for values of x in the neighbourhood of x_0 and diverge for $|x-x_0| > \in$ while at the end points $x_0 \pm \in$, it may either converge or diverge

(iii) Absolutely for all x, i.e. for $-\infty < x < \infty$

The set of Values of x for which the power series converges is said to be the interval of convergence and it is denoted by ***I***. Incase power series converges on an interval I: $|x-x_0|<R$ being a positive constant, then the power series defines a function f(x) which is continuous for each x in ***I***,. If f(x) is a function defined by a power series, then,

$$f(x) = C_0 + C_1(x-x_0) + C_2(x-x_0)^2 + \ldots + C_n(x-x_0)^n + , \text{ for } \boldsymbol{I}: |x-x_0|<R$$

2.2 Taylor Series Method

The Taylor series was formally introduced by the English Mathematician-Brook Taylor in 1715 as reported in Irene (1970). If the series is centred at zero, the series is called a Maclaurin series named after the Scottish Mathematician. It is a representation of a function in an infinite sum of terms that are calculated from the values of the function's derivatives at a single point.

Let $y = f(x)$ be a solution of the equation

$$\frac{dy}{dx} = f(x,y), y\,(x_0) = y_0$$

Expanding it by Taylor Series about x_0, we get

$$f(x) = f(x_0) + \frac{f^1(x-x_0)}{1!}(x_0) + \frac{f^{ll}(x-x_0)}{2!}(x_0)^2 + \cdots + \frac{f^n(x-x_0)}{n!}(x_0)^n \qquad 2.2.1$$

Or

$$y(x) = y_0 + \frac{(x-x_0)}{1!}y_0{}^{l} + \frac{(x-x_0)^2}{2!}y_0{}^{ll} + \frac{(x-x_0)^3}{3!}y_0{}^{lll} + \cdots + \frac{(x-x_0)^n}{n!}(y_0)^n \qquad 2.2.2$$

2.3 Ordinary Differential Equation (ODE)
An ordinary differential equation (ODE) is an equation involving derivatives with respect to a single independent variable.

Example $\frac{dy}{dx} = f(x, y)$

2.4 Order
The order of a differential equation is defined as the order of the highest derivative. Example, $M\frac{d^2x}{dt^2} + kx = f(t)$, order is 2;

2.5 Mathematical Problem
Consists of a set of equations in which you will find some known variables and some unknown variables.

2.6 Scientific Computing
Solving the mathematical problem on the computer for the unknown variables.

2.7 Numerical Analysis
It is the branch of mathematics involved with scientific computing for obtaining numerical values (solutions) to various mathematical problems for which there are always physical (real-life) applications. It involves the study, development and analysis of algorithms.

2.8 Numerical Treatment
Numerical treatment of a mathematical problem is the numerical solution (remedy) to a mathematical problem. The accuracy of the numerical method is indicated by the order of the error term. The higher the order the better the accuracy.

3.0 THEORY INTERRELATING STABILITY, CONSISTENCY AND CONVERGENCE

3.1 Well-Posedness Condition
Suppose that f and f_y, its first partial derivative with respect to y, are continuous for x in [a,b]. Then any of the initial or boundary value problems of the following:

$$y' = f(x,y),\ x \in [a, b]; \quad y(a) = y_0 \qquad 3.1.1$$

$$y'' = f(x,y),\ x \in [a, b]; \ y(a) = \eta,\ y(b) = \beta \qquad 3.1.2$$

$$y'' = f(x,y),\ x \in [a, b]; \ y(0) = y_0,\ \ y'(0) = \eta_0 \qquad 3.1.3$$

have unique solutions $y(x)$ for $a \leq x \leq b$ and the problem is said to be well-posed (See Burden & Faires 1993). We shall, wherever appropriate, mention some **contributions of late Prof. S. O. Fatunla (1943-1995)** on whose theory of block methods we rely for convergence analysis of all the proposed block methods. For us to keep pace with change, and meet the challenges of the future, the algorithms were proposed in a way to circumvent some identified draw backs of the Linear Multi-step Methods (LMMs) belonging to the general class of optimal order block methods developed by Fatunla (1992; 1994).

3.2 Existence of Solution

Let it be that

- $f(x, y(x))$ is continuous in the interval $[a,b] \subset \Re$
- $f(x, y(x))$ is lipschitzian with regard to its second argument y:

  that is, L is called the lipsschitz constant,

  $\|f(x, y(x)) - f(x, Z(x))\| \leq L \, \| y(x) - Z(x)\|$

For all $\forall\ x \in\ [a, b]; Z(x), y(x) \in S \subset \Re$.

And the solution, $y(x) \in \Re$ of the ODEs (3.1.1) to (3.1.3) exists and is unique in the strip

$S = \{[a, b] \text{ x } \{y(x) : \left| y(x) \right| < \}\}$

Then the solution, y(x) ER or the ODEs ((3.1.1) to (3.1.3)) exists and is unique in the strip

S = {*a,b x{y x;* |y(x)<….} schematically, the picture of S is glaring in Fig. 3.2.1.

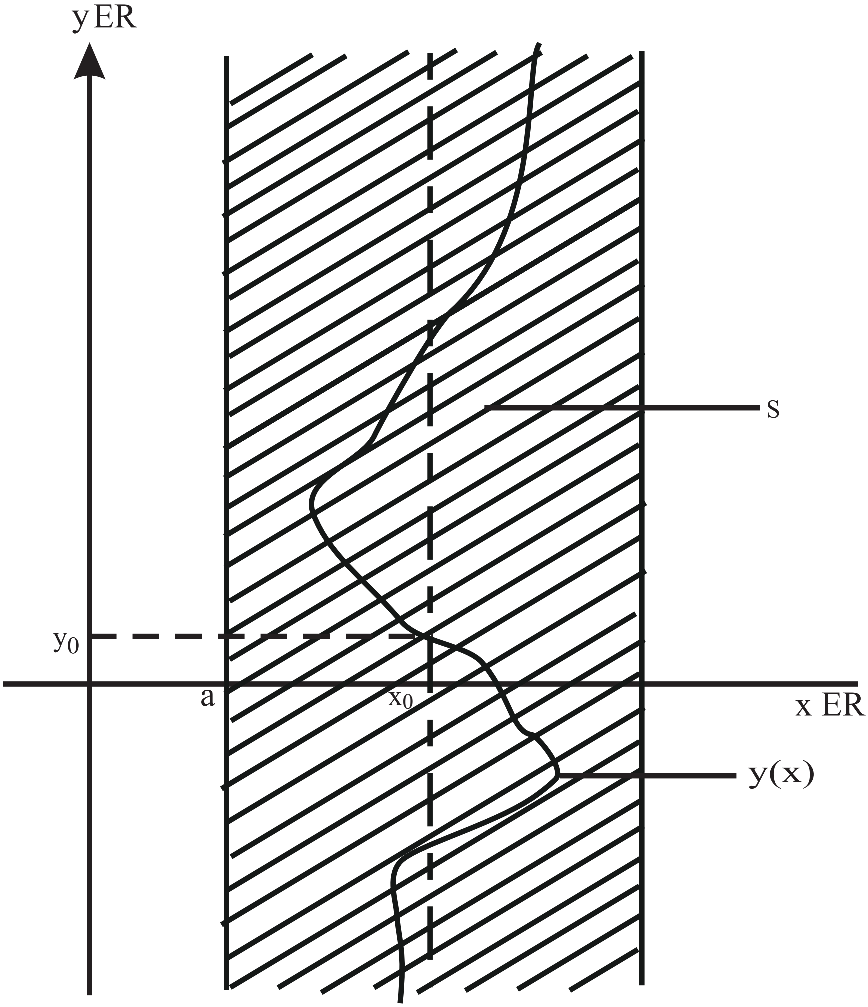

Figure: 3.2.1

3.3 Uniqueness of Solution

Suppose p(x), q(x) and g(x) are continuous on an interval (a, b) which contains the point x_0. Then for any choice of the initial value y_0 and y_1, there exists a unique solution y(x) on the whole interval (a, b) to the Initial Value Problems (IVP).

$$y'' + p(x)\,y' + q(x)y = g(x) \qquad 3.3.1$$

$$y(x_0) = y_0,\ y'(x_0) = y_1$$

The proof of this theorem can be found in Kentnagle and Edward (1994).

3.4 Problem Formulation

In this section we consider the construction of multi-step collocation method (MC) for constant step size h, although the step size h can be a variable, it is assumed throughout the *lecture as constant, for simplicity*.

The basic method seeks a k-step (k > 0) expansion of the form

$$y(x) = \sum_{r=0}^{p-1} a_r Q_r(x) \qquad x \in [x_n, x_{n+k}] \qquad 3.4.1$$

Where a_r are constants and Qr(x), r = 0,…, p-1 are the Taylor series constants.

This is based on a matrix inverse and an arbitrary basis vector function for the ODEs of the form

$$Ly = f(x, y),\ a \le x \le b$$

And

$$L \equiv \left(\frac{d}{dx}, \frac{d^2}{dx^2}, \ldots, \frac{d^N}{dx^N} \right)$$

Depending on the problem of interest, where y satisfies initial or boundary conditions.

Let us consider the second order system of ODEs type (3.1.2-3.1.3),

$$y'' = f(x, y), \qquad a < x < b,\ y, f \in R^S \text{ i.e.}$$

$$Ly \equiv \frac{d^2 y}{dx^2} = f(x, y), a \le x \le b$$

where y satisfies a given set of associated conditions, which are either all initial, all boundary or mixed conditions. The idea of the k-step (MC), following Awoyemi (2002; 2001) and Onumanyi *et al.* (1994) in Yahaya (2004) is to find a polynomial U of the form

$$U(x) \cong y(x) = \sum_{j=0}^{t-1} {}''{}_j(x) y_{n+j} + h^2 \sum_{j=0}^{m-1} {}''{}_j(x)\ f(\overline{x}_j, y(\overline{x}_j)), \quad -$$

$$x_n \leq x \leq x_{n+k} \qquad 3.4.2$$

Where t denotes the number of interpolation points

$\overline{x}_{n+i}$, i = 0,1,…, t-1,

and m denotes the number of distinct collocation points

$\overline{x}_j \in [x_n, x_{n+k}]$, j = 0, 1,…, m-1

The point $\overline{x}_i$; $\overline{x}_i$ are chosen from the step points x_{n+i} as well as from one or more off-step points.

Now, consider the unknowns a_r in (3.4.3) in the form.

$$a_r = \sum_{j=0}^{t-1} {}''{}_{r+1,j}\ y_{n+j} + h^2 \sum_{j=0}^{m-1} {}''{}_{r+1,j}\ f(\overline{x}_j, y(\overline{x}_j)) \qquad 3.4.3$$

r = 0, 1,…,p-1, where p = t + m.

Let us denote $f(\overline{x}_j, y(\overline{x}_j))$ by f_{n+j}, j = 0,1,…,m-1 in (3.4.3) and the real constants $\Phi_{r+1,j}$, $h^2\psi_{r+1,j}$ are to be determined as the elements of a matrix of dimension P x P of the form

$$C=\begin{pmatrix} \phi_{1,0} & \phi_{1,1}\cdots & \phi_{1,t-1} & h^2\psi_{1,0} & h^2\psi_{1,1\ldots} & h^2\psi_{1,m-1} \\ \phi_{2,0} & \phi_{2,1\ldots} & \phi_{2,t-1} & h^2\psi_{2,0} & h^2\psi_{2,1\ldots} & h^2\psi_{2,m-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \phi_{i+1,0} & \phi_{i+1,1\ldots} & \phi_{i+1,t-1} & h^2\psi_{i+1,0} & h^2\psi_{i+1,1\ldots} & h^2\psi_{i+1,m-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \phi_{p,0} & \phi_{p,1\ldots} & \phi_{p,t-1} & h^2\psi_{p,0} & h^2\psi_{p,1\ldots} & h^2\psi_{p,m-1} \end{pmatrix} \qquad 3.4.4$$

If we introduce the following vector notations:

$$\underline{Q}(x) = (Q_0(x), Q_1(x), \ldots, Q_{p-1}(x))^T,$$

$$\underline{F} = (y_n, y_{n+1}, \ldots, y_{n+t-1}, f_n, f_{n+1,} \ldots f_{n+m-1})^T$$

Where T denotes "transpose of" and

$$\underline{a} = (a_0, a_1, \ldots, a_{p-1})^T.$$

Then (3.4.1) can be written in a vector form

$$y(x) = \underline{a}^T \underline{Q}(x) . \qquad 3.4.5$$

Let y(x) satisfy the conditions

$$y(x_j) = y_j,\ j = n, n+1, \ldots n + t-1 \qquad 3.4.6a$$

$$Ly(\bar{x}_j) = f_{n+j}, j = 0,1,\ldots,m-1 \qquad 3.4.6b$$

This leads to the single set of algebraic equations

$$\sum_{r=0}^{p-1} a_r \begin{bmatrix} Q_r & (x_{n+j}) \\ LQ_r & (\overline{x_j}) \end{bmatrix} = \begin{pmatrix} y_j, j = n, n+1, \ldots, n+t-1 \\ f_{n+j}, j = 0, 1, \ldots, m-1 \end{pmatrix}$$

And expressed in a matrix form as follows:

$$\underline{\underline{M}} \quad \underline{a} = \underline{F} \qquad 3.4.7$$

$$\underline{a} = \underline{\underline{M}}^{-1} \underline{F} \qquad 3.4.8$$

$$a^{T} = (\underline{\underline{M}}^{-1} F)^{T} = F^{T} (M^{-1})$$

Where M is the non-singular matrix of dimension P x P of the form

$$\underline{\underline{M}} = \begin{bmatrix} Q_0(x_n) & .... & Q_{\rho-1}(x_n) \\ Q_0(x_{n+1}) & .... & Q_{\rho-1}(x_{n+1}) \\ . & .... & . \\ Q_0(\overline{x}_{n+t-1}) & .... & Q_{\rho-1}(\overline{x}_{n+t-1}) \\ L[Q_0(\overline{x}_0)] & .... & L[Q_{\rho-1}(\overline{x}_0)] \\ L[Q_0(\overline{x}_1)] & .... & L[Q_{\rho-1}(\overline{x}_1)] \\ . & .... & . \\ L[Q_0(\overline{x}_{m-1})] & .... & L[Q_{\rho-1}(\overline{x}_{m-1})] \end{bmatrix} \qquad 3.4.9$$

Then, (3.4.5) becomes explicitly given by the following

$$y(x) = \underline{F}^{T} (\underline{\underline{M}}^{-1})^{T} \underline{Q}(x), \; ; \; x \in [x_n, x_{n+k}]; \qquad 3.4.10$$

which is the desired (continuous interpolant) interpolation formula for solving (3.1.2 and 3.1.3)

My problem of interest in this section, is given by (3.1.1 to 3.1.3). Where f satisfies the conditions for the existence and uniqueness of y. It must be remarked here that as attractive as this result (3.4.10) is, it is not economical to use it in the step-by–step numerical integration due to the cost of evaluating matrix inverses of large size at every step of integration. Rather see Yahaya (2004) as implemented in Yusuph and Onumanyi (2002).

3.5 Existence and Uniqueness of M^{-1}

It is done once and y(x) is obtained as in (3.5.1), (3.5.2) below. But from (3.4.5),

let

$Q_r(x) = x^r$, r = 0, 1,…, P-1

$$y(x) = \sum_{r=0}^{p-1} \left(a_r x^r\right) \qquad 3.5.1$$

where a_r is given by (3.4.3).

$$y(x) = \sum_{j=0}^{t-1} \Phi_j(x) y_{n+j} + h^2 \sum_{j=0}^{m-1} \Psi_j(x) f_{n+j} \qquad 3.5.2$$

$$\Phi_j(x) = \sum_{r=0}^{p-1} \Phi_{r+1,j} x^r; \;\; h^2 \Psi_j(x) = \sum_{r=0}^{p-1} h^2 \Psi_{r+1,j} x^r \qquad 3.5.3$$

and $\Phi_{r+1,j}$, $h^2\psi_{r+1,j}$ are stored as the elements of the matrix $\underline{\underline{C}}$ given by (3.4.4). Observe that (3.5.2), (3.5.3), give a continuous LMM whose discrete counterpart is given by
$y(x_{n+k}) = y_{n+k.}$

From (3.4.3), we can express a_r as follows:

$$a_r = (\Phi_{i+1,0}, \Phi_{i+1,1}, \ldots, \Phi_{i+1,t-1}, h^2\psi_{i+1,0}, h^2\psi_{i+1,1}, \ldots, h^2\psi_{i+1,m-1})\underline{F} \qquad 3.5.4$$

$$a_r = \underline{C}_{i+1} \underline{F}, i = 0,1,\ldots, p-1, r = 0,1,\ldots, p-1 \qquad 3.5.5$$

Where $\underline{C}_{i+1}$ denotes $(i + 1)^{th}$ row vector of $\underline{C}$ in (3.4.4).

Hence from (3.5.5)

$$\underline{a} = \underline{\underline{C}}\ \underline{F} \qquad 3.5.6$$

By the existence and uniqueness of solution y(x) to (3.4.3 and 3.4.4) and comparing (3.5.6) with (3.4.8) we get the desired result that

$$\underline{\underline{C}} = \underline{\underline{M}}^{-1} \qquad 3.5.7$$

Next to show that $\underline{\underline{M}}$ is non-singular. Suppose $\underline{\underline{M}}$ is singular, then $\ni \underline{a} \neq 0$, such that

$$\underline{\underline{M}}\ \underline{a} = 0 \qquad 3.5.8$$

Involving t + m equations in t + m undetermined constants.

That is $\ni$

$$\underline{a} = (a_0, a_1, \ldots a_{t-1}, a_t, a_{t+1}, \ldots, a_{t+m-1})^T \qquad 3.5.9$$

Such that we have a polynomial $p_{t+m-1}(x)$ of the form

$$P_{t+m-1}(x) = a_0 + a_1 x + \ldots + a_{t+m-1} x^{t+m-1} \qquad 3.5.10$$

of degree = t + m-1, and satisfying (3.5.8).

This is equivalent to $P_{t+m-1}(x)$ in (3.5.10) having t+m roots by

$P_{t+m-1}(x) = 0$; i = n, n+1, …., n + t - 1 (t Zeroes)

$LP_{t+m-1}(\bar{x}_j) = 0$; j = 0,1, ……, m-1 (m additional Zeroes);

$$L = \frac{d^2}{dx^2}$$

But $P_{t+m-1}(x)$ having t + m roots is absurd because a polynomial of degree = t + m-1, can have at most t + m-1 roots. Hence, my earlier assumption that $\underline{\underline{M}}$ is singular is **wrong**. Thus, $\underline{\underline{M}}$ is non-singular and M^{-1} exists and is unique. It can be obtained by MAPLE a sub-menu in MATLAB or any known matrix inverse algorithm. Mr. Vice-Chancellor Sir, I want to emphasize the fact that derivation formula (3.4.10) is an explicitly determined polynomial of degree p-1, which is valid in $[x_n, x_{n+k}]$ for any chosen polynomial basis Q(x). When simplified as shown in (3.5.2), (3.5.3), it can be easily seen as multi-step collocation interpolant whose step number is k. The way of implementation of (3.4.10) determines if it will overlap or not along the integration process. By evaluation (interpolation and extrapolation) of y(x) in (3.4.10) at mesh and off-step, as well as its second derivative, in line with the proposed solution approach, one can obtain sufficient multiple FDMs for a simultaneous solution (block) to avoid overlap of subintervals.

3.6 A Link with Interpolation

Consider the polynomial interpolation problem involving

$$y^I(x) = a_0 + a_1 x + \ldots + a_{t-1} x^{t-1}, \qquad x_n = x = x_{n+k} \qquad 3.6.1$$

satisfying

$$y^I(x_i) = y_i, \qquad i = 0, n, n+1, \ldots, n+t-1, \qquad 3.6.2$$

leads to

$$\underline{\underline{V}}\,\underline{a} = \underline{F} \qquad 3.6.3$$

$$\underline{a} = (a_0, a_1, \ldots, a_{t-1})^T$$

$$\underline{F} = (y_n, y_{n+1}, \ldots, y_{n+t-1})^T$$

And $\underline{\underline{V}}$ is a vandermonde's matrix of t x t dimension contained in $\underline{\underline{M}}$ by the first t rows and t columns.

Hence,

$$\underline{a} = \underline{\underline{V}}^{-1} \underline{F} \qquad 3.6.4$$

Since $\underline{\underline{V}}$ is known to be non-singular. Thus,

$$y^I(x) = \underline{a}^T \underline{Q}^I(x)\,;\, \underline{Q}^I(x) = (1, x, \ldots, x^{t-1})^T$$

$$= \underline{F}^T (V^1) \underline{Q}^I(x) \qquad 3.6.5$$

Apart from different dimensions and the enlargement of $\underline{\underline{V}}$ to $\underline{\underline{M}}$ by additional derivative parameters, formula (3.6.5) is practically the same form as formula (3.4.10), which shows the link.

3.7 Theorem (Equivalent Representation Result)

The form (3.4.2) for $y(x)$ is equivalent to the power series form

$$y(x) = a_0 + a_1x + a_2x^2 + \ldots + a_{t+m-1}x^{t+m-1} \qquad 3.7.1$$

Where a_i, $i = 0,\ldots,t+m-1$ are undetermined constants.

Proof:

Starting with (3.4.2) and (3.4.3) we obtain, by rearranging the terms of (3.4.2),

$$y(x) = \sum_{i=0}^{t+m-1} a_i x^i$$

$$a_i = \sum_{j=0}^{t-1} \Phi_{j,i+1} y_{n+j} + \sum_{j=0}^{m+1} \Psi_{j,i+1} - f(\bar{x}_{j+1}, y(\bar{x}_{j+1}))$$

This yields (3.7.1).

Conversely, if we start with (3.7.1) using the interpolation and collocation conditions stated below:

$$\left.\begin{aligned} &\Phi_j(x_{n+i}) = \delta_{i,j}, \ j = 0,\ldots,t-1; \ i = 0,\ldots,t-1 \\ &h^2\psi_j(x_{n+i}) = 0, \ j = 0;\ldots, m-1; \ I = 0,1,\ldots m-1 \end{aligned}\right\} \qquad 3.7.2$$

and

$$\left.\begin{aligned} &\Phi_j''(\bar{x}_i) = 0, \ j = 0,1,\ldots\ldots, t-1; \ I = 0,\ldots\ldots,m-1 \\ &h^2\psi_j(\bar{x}_i) = \delta_{i,j}, \ i = 0,1,\ldots\ldots,m-1 \ ; \ i = 0,1,\ldots, m-1 \end{aligned}\right\} \qquad 3.7.3$$

and with notations

$a = (a_0, a_1 \ldots, a_{t+m-1})^T$,

$\underline{F} = (y_n,\ldots,y_{n+k-1}, f(\bar{x}_1, y(\bar{x}_1)),\ldots,f(\bar{x}_m, y(\bar{x}_m)))^T$

We have that

$$D\underline{a} = \underline{F}$$

Where D, for this case is given by

$$
D = \begin{bmatrix}
1 & x_n & (x_n)^2 & \cdot\ \cdot & \cdot & (x_n)^{t+m-1} \\
1 & x_{n+1} & (x_{n+1})^2 & \cdot\ \cdot & \cdot & (x_{n+1})^{t+m-1} \\
\cdot & \cdot & \cdot & \cdot\ \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot\ \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot\ \cdot & \cdot & \cdot \\
1 & x_{n+t-1} & (x_{n+t-1})^2 & \cdot\ \cdot & \cdot & (x_{n+t-1})^{t+m-1} \\
0 & 0 & 2 & \cdot\ \cdot\cdot & (t+m-1)\cdot(t+m-2) & (\bar{x}_0)^{t+m-3} \\
\cdot & \cdot & \cdot & \cdot\ \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot\ \cdot\cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot\ \cdot & \cdot & \cdot \\
0 & 0 & 2 & \cdot\ \cdot & (t+m-1)\cdot(t+m-2) & (\bar{x}_{m-1})^{t+m-3}
\end{bmatrix} \quad 3.7.4
$$

and T denotes "transpose of" ;

Thus, $\underline{a} = D^{-1}\ \underline{F}$ 3.7.5

By the uniqueness of D^{-1}, we get

$a = cF$ 3.7.6

and by rearrangement of terms (3.7.6) leads to (3.4.1)

Thus, complete the proof of theorem 3.7

Corollary:

$$y(x) = F^T\ \underline{\underline{C}}^T\ \underline{P}(x),$$

with

$$\underline{P}(x) = (1, x\ldots, x^{t+m-1})^T$$

<u>Proof:</u>

From (3.7.1), it follows that

$$y(x) = a^T\ P(x) = F^T C\ \ P(x).$$

From these theoretical derivations, therefore, many important classes of Linear Multistep Methods (LMM) and Runge-Kutta Methods (RKM) were re-produced, including new ones, which are generally more accurate with adequate absolute-stability intervals for non-Stiff Problems. The use of Power series as basis function in the assumed trial solution was exploited. Although, the Analytic solution y is assumed to exist and unique in [a, b], use of other low order piece-wise polynomials, like Splines, exponential functions, logarithm functions and rational functions for special applications as may be appropriate could be considered. This work laid the foundation for research work in this area for several authors thereafter

4.0 LINEAR MULTISTEP METHODS

The problem of approximating a function is of great significance in Numerical Analysis due to its importance in the development of software for digital computers. Although, Numerical Analysis can be considered as the study of obtaining approximations to solution of mathematical problems (by arithmetic process), the need to approximate non-arithmetic quantities and to ascertain the errors associated with such approximations lies therefore at the heart of much of numerical analysis. The numerical solution of differential equations is currently central to numerical analysis. Numerical methods for obtaining solutions to the class of problems ((3.1.1) to (3.1.3)) have received great attention from researchers. Much fruitful work has been done by skilful analyst to devise specific one-step and linear multistep methods that are highly accurate, stable, and convenient to use computationally. See Yahaya (2004); Onumanyi *et al*. (2002; 1999).

The traditionally successful numerical methods such as Adams-moulton and the explicit Runge-Kutta suffered step size constraint imposed by stability when applied to stiff problems. The modelling of some physical phenomena in mechanical, biological, chemical system, etc, may result in second order IVP of the form.

$$y'' = f(x,y),\ x \in [a, b];\ y(0) = y_0,\ \ y'(0) = \eta_0$$

In which the first derivative is absent. In fact, these systems often occur in mechanical systems without dissipation, satellite tracking, warning systems and celestial mechanics. Their theoretical solutions are usually periodic as reported in Fatunla *et al* (1999). In this lecture, I adopt the usual convention. Collocation solutions are desirable from both practical and theoretical considerations and their advantages are now creating growing interest in continuous integration algorithms for the numerical solution of ODEs. In particular, collocation solutions of the ODEs by their nature are continuous.

4.1 Definition of Linear Multistep Method (LMM)

Unlike the one-step method that utilizes one previous value of the numerical solution to approximate the subsequent value, a K-Step multistep method utilizes (k-1) previous value.

A general k-step LMM is given in the form

$$\sum_{j=0}^{k} \alpha_j (x) y_{n+j} = h^{\mu} \sum_{j=0}^{k} \beta_j (x) f_{n+j} \qquad 4.1.1$$

where μ is the order of differential Equation in consideration.

4.2 Definition of Block Method

A block method is formulated in linear multi-step method form. Yahaya (2004) explained that block method preserve the advantage of one-step methods, of being self-starting and permitting easy change of step-length. A general k-block, r-point block method is a matrix of finite difference equation

With the r-vector Y_q and F_q (for n = qr, q = 1, 2, …). Specified as

$$Y_q = \begin{pmatrix} y_{n+1} \\ y_{n+2} \\ \cdot \\ \cdot \\ \cdot \\ y_{n+r} \end{pmatrix}, \qquad F_q = \begin{pmatrix} f_{n+1} \\ f_{n+2} \\ \cdot \\ \cdot \\ \cdot \\ f_{n+r} \end{pmatrix} \qquad 4.2.1$$

The s-block r-point methods for (3.1.1- 3.1.3) are given by the matrix finite difference equation

$$-A^{(0)} y_q = \sum_{i=1}^{s} {}^{(i)} Y_{q-i} + h^2 \cdot \sum_{i=0}^{s} B^{(i)} f_{q-i} \qquad 4.2.2$$

where $A^{(i)}$, $B^{(i)}$, I = 0 (1) s are r by r matrices respectively with element $a^{(i)}_{l,j}$, $b^{(i)}_{l,j}$ for l, j = 1(1)r, and $Y_0 = \begin{pmatrix} y_0 \\ h \cdot z_0 \end{pmatrix}$, q = 0 is given with the IVPs.

For the hybrid methods in section (4.3), r takes both integer and non-integer values. The block schemes (4.2.2) is explicit if the coefficient matrix $B^{(0)}$ is a null matrix.

Let

$$Z_q = \begin{pmatrix} y(x_{n+1}) \\ y(x_{n+2}) \\ \cdot \\ \cdot \\ \cdot \\ y(x_{n+r}) \end{pmatrix} \qquad 4.2.3$$

represent the theoretical solution to (3.1.1 - 3.1.3). Most known block methods belonging to the class (4.2.2) are s = 1, in my later years I developed a class of s = 2 for c^1solution space. See Yusuph and Onumanyi (2002) A'; s and B'; s are properly chosen "r x r" Matrix coefficient; when n- is the order of differential equation in consideration, and m = 0,1,2,… represent-the number n = mr is the first step number of the mth block and r is the proposed block size.

4.3 Hybrid Methods

Linear Multistep Methods (LMM), though generally the more efficient in terms of accuracy and weak stability properties for a given number of function evaluations per step, suffered the disadvantage of requiring additional starting values and special procedures for changing step length. These difficulties would be reduced, without sacrifice, if we could lower the step number of a linear multistep method without reducing its order. The difficultly here lies in satisfying the essential condition of zero-stability. As stated earlier, an implicit linear K-step method is capable of achieving order 2k, but if the condition of zero stability is to be satisfied then the maximum attainable order is only K+2 when K is even and K+1, when K is odd. This Zero stability barrier was circumvented by the introduction of some function evaluation at off-step points. Of note, is that this method - Hybrid retained certain Linear multistep characteristics but share with Runge-Kuta Methods the property of utilizing data at points other than the step points $[X_n \,/\, X_n = a+nh]$

A K-Step hybrid formula to be of the type

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j} + h\,\beta_v f_{n+v} \qquad 4.3.1$$

Where a_0 and β_0 are not both zero, $v \notin (0,1,2,\ldots.k)$ and $\alpha_k = +1,$ to remove arbitrariness. Notable methods here, include those in Yahaya (2004), Yahaya and Adegboye (2011); Yahaya *et al.,* (2010);

Yahaya and Sokoto (2010); Badmus and Yahaya (2009); Yahaya and Sagir (2013) to mention just a few. Despite the fact that these methods allow the freedom of by-passing the barrier identified by Dahlquist and Butcher, they unfortunately inherit some of the misgivings of their source methods.

4.4 Parallel Methods

The emergence of parallel methods has been triggered by the introduction of vector and parallel computers since the second half of 1970's and they are currently attracting serious research attention as mentioned earlier because of their efficiency by virtue of parallelism. A parallel block method is of the form

$$y_m = \sum_{j=o}^{k} A^{(j)}\, y_{m-j} + h \sum_{j=o}^{k} \beta^{(j)}\, f_{m-j} \qquad 4.4.1$$

For first order ODEs types in equation (3.1.1)

$$y_m = \sum_{j=o}^{k} A^{(j)}\, y_{m-j} + h^2 \sum_{j=o}^{k} \beta^{(j)}\, f_{m-j} \qquad 4.4.2$$

For special second order ODES designated by (3.1.2 and 3.1.3) where in both cases (4.4.1) and (4.4.2)

$$y_{m-j} = (y_{n-js+1}, y_{n-js+2}, \cdots\cdots\, y_{n-js+s})^T \qquad 4.4.3$$

$$F_{m-j} = (F_{n-js+1}, F_{n-js+2}, \cdots\cdots\, F_{n-js+s})^T, n = ms$$

The matrix finite difference scheme $(4.4.1 \propto 4.4.2)$ is an s-block r-point parallel method. This is an automated reformulation of the conventional LMM (4.4.1 to 4.4.3), as proposed in Fatunla (1991; 1992) it is thus possible to assign r processors to r different time levels at s different blocks defined in (4.4.2) for the numerical integration of the class of problems (3.1.1to 3.1.3) computational speed up is achievable with these methods because of the admissible parallelism across time. Despite the potentials of these methods, their major impediments include: inherent problems of inter communication of information between different processors in process, automatic step-size control, automatic error control and estimation and stability constraints as noted by Gear (1987a) and Fatunla (1991). Furthermore, Gear (1987b) in his survey paper remarked that parallelism across space could be effective for very large scale parallelism while parallelism across time appears to be more appropriate for small scale, parallelism for an SIMD or MIMD Machine can be employed. Relatively, few parallel methods for the integration of our class of problems and other methods, which are parallizable, as seen in Section 4.4 of this lecture, have been proposed compared to their sequential counter parts.

4.5 Runge –Kutta Methods

In the past few years, computational mathematics has seen a steady growth. This is partly due to high speed digital computation being more readily available and partly because so many disciplines now find mathematics an essential element of their curriculum. Students of physics, chemistry, biology, engineering and economics, are concerned with mathematics and computing mainly as tools, but tools with which they must acquire proficiency. On the other hand, courses for mathematicians must take cognizance of the existence of electronic computers. Recall the initial value problem.

$$y' = f(x,y) \quad ,y(a) = \sigma$$

The easiest computational methods to implement is Euler's rule

$$y_{n+1} - y_n = hf\left(x_n, y_n\right) = hf_n \_\_\_\_\_\_\_\_ \quad (1)$$

It is explicit and being a one-step method.

It requires no additional starting values and readily permits a change of step length during the computation. R-K method is concerned with the determination of solution to a class of problems in Ordinary Differential Equations (ODEs).

4.6 Dalhquist Order Barrier for Linear Multistep Methods (LMM)

a. A zero-stable K-step LMM is of maximum order p, with

$$p = k + 1, when\ \ K\ is\ odd\ and\ k + 2\ , when\ k\ is\ Even$$

b. An explicit LMM can not attain A-stable if the step number K is such that K>2

c. The order P of an A-stable LMM can not exceed two. In fact, the Trapezoidal rule which is of order P = 2 with step number K = 1, known for its A- stability has the smallest error constant of $C^* = {}^{1}/_{12}$.

Remarks

These barriers have a profound effect on the development of LMM and explicit RKM. These barrier Theorems have given a clearer vision on the limited usefulness of explicit RKM and LMM. It has provided a very useful guidance on the construction of new methods. In fact, this has engendered a quasi-variation of the above classes of methods. The idea actually is the hybridization of these classes of methods with the deliberate intention to by-pass the limitations imposed by these barriers. The need for methods of high order and superior stability properties become imperative considering the fact that the numerical experiences have shown that some stiff problems of special second order ODEs especially from chemical kinetics and semi- discretisation of partial differential equations demands as such.

5.0 MY HUMBLE CONTRIBUTIONS

During my first degree in Mathematics, project writing was optional before my set. By God's design, the Department under the leadership of Prof. M. A. Ibiejugba made it compulsory and I was assigned to Mr. S.T. Oni (who later moved to FUT Akure and became a Professor there). We worked on beams subjected to moving forces, investigating effect of damping on the transverse deflection, in the field of dynamics. When I eventually found myself in University of Jos for postgraduate studies, we did several courses that covered the three major areas of mathematics, namely Pure, Applied and Analysis. I specialized in Numerical Analysis with special interest in solving differential equations, be it ordinary or partial, using linear multistep methods or Runge-Kutta methods. I have tried not to go into the world of Analysis, though the temptation has been so fierce. In my own attempt to improve the area of Numerical Analysis for the benefit of Mankind, I will discuss three broad areas my contributions covered. Lie and Norsett (1989) were the first to study multi-step collocation with super convergence results. In order to characterize a multi-step collocation method as a multi-step Runge-Kutta. They proposed a K-step collocation method with M distinct collocation points whose continuous coefficients were specified explicitly through integration of some Lagrange type interpolants. However, two restrictions were noted in their formulations, namely, all the K-previous mess points must be used for interpolation while all M collocation points are restricted to the last Step (x_{n+k-1}, x_n+k).

As a result of these limitations only the one leg method of Dahlquist (1979; 1983) and the backward differentiation formulae methods, which involve only one-collocation point and K-interpolation points were recovered from their formulation. This clearly indicates the lack of generality of their approach. Mr. Vice-Chancellor Sir, this is where my journey begins! I want to inform this gathering that, my attempt to avoid cumbersome integration process and also allow for flexibility in the choice of interpolation and collocation points truly yielded results as you would see in the subsequent sections.

Ladies and gentlemen, **Do I say: "The choice is yours" or to be in tandem with today's topic "Your Opinion is Yours"**. Indeed, it is for you and for me to decide not be ignorant because an ignorant person is a **gamble**r! There is this Gambler's Fallacy or Monte-Carlo Fallacy; which states that *"The notion that the Probability of the occurrence of an event increases with the length of time since it initially occurs"!* This in itself is euphemism for Gambler's ruin game! The results of gambling have not been rewarding in any way. We therefore need to seek after knowledge. Some facts are already known or written and can be obtained in relevant publications. It is thus necessary to read very widely and search for facts. Other facts are hidden and should be searched out and identified through properly guided researches. This is the motivation for my research activities on the subject of discussion today.

There are three main directions in which research is channelled presently, namely; the modelling and simulation group i.e. studies that look at real-life problems viz the rate at which a variable changes and is modelled as ODEs. The abstract and classical analysis group; where they consider an ODE with specific unknown parameters or unknown coefficients of the ODE terms, subject the ODE to certain constraints and then obtain conditions or forms under which the coefficients or parameters can be determined. Most current research work in classical mathematics to which ODE belongs have been in existence since 1642 AD; that is over three centuries! It is not therefore a surprise that this area is the richest and even tending towards abstraction. The third group is the computerized group and this is concerned with the development, analysis testing and implementation of methods to solve problems in ODEs. This group handles effectively problems whose solutions cannot be established with ease through abstract and classical methods but to evaluate them without computational methods becomes extremely difficult. These are the main reasons why in today's world of ODEs, emphasis is drifting to computerized mathematics. To this end, we have several ODE solvers, and software, e.g. Matlab, etc. My research on the Runge-Kutta methods was directed towards developing continuous schemes both for existing methods and new schemes. As will be discussed later, the novelty of my contribution is in the reformulation of some existing LMM into R-K method and vice-versa. Yahaya and Adegboye (2011) illustrated the direct application of collocation approach. The method yields a continuous two-step hybrid multistep method which was reformulated into the 5-stage well known Runge-Kutta method using the Butcher analysis. Interestingly the method produced simultaneously approximation of the solution of both linear and nonlinear initial value problem at a block of two points x_{n+1} and

x_{n+2} Butcher (2006) added that both R-K methods and LMMs have limitations and the class of GLMs offer new possibilities of constructing new formulas which attempt to combine the advantages of R-K methods (large regions of stability) and LMMs (high stage order) at the same time avoiding the disadvantages of these methods. Yahaya and Ajibade (2010) presented a report of research on a reformulation of a two-step hybrid linear multistep method (LMM) into a 3-stage Runge-Kutta type (RKT), which was employed to solve first order IVPs. In an earlier work of Yahaya and Adegboye in 2007. The authors constructed and implemented a new Quade's type four-step block Hybrid multi-step method for accurate and efficient parallel solution of first order ODEs. The results converged better to the exact solution with A-stable region of absolute stability in view of this property. Yahaya and Adegboye in 2011 reformulated the Quade's type four step Block Hybrid multistep method into single-step Runge-kutta method for solution of both first and second order ODEs. Interestingly the single-step turned-out to be of order six and A-stable, with the R-K stability property and has an implicit structure for efficient implementation and that enabled us to handle 3-classes of problems -first order IVP and second order (special or general) (see the paper for details).

I observe that the problem of General second order ordinary differential equations is not commonly discussed in literature. I wish to state that my contributions in R-K method was made possible because it was a multistage and LMM are multi-value procedure, since my wish is to have interrelated aims in a single method such as high order, low error constants, satisfactory stability property, low cost of implementation and above all self –starting. Yahaya and Adegboye in 2012 explored this and extended the symmetric implicit Runge-Kutta method for the integration of first order ODEs to a symmetric implicit super Runge-Kutta Nystrom method for direct integration of General third order Initial Value Problems (IVPs). The theory of Nystrom method was adopted in the derivation. The implicit structure allows for efficient implementation and produce simultaneously approximation of the solution. Although it is possible to integrate a third order IVP by reducing it to first order system and apply one of the methods available for such system, it seems more natural to provide commercial method in order to integrate the problem directly. The advantages of these approaches presented in the paper lie in the fact that they are able to exploit special information about ODEs and this results in efficiency increase i.e High accuracy at low cost. A basic property which any acceptable numerical method must possess is that (y_n) generated by the schemes converges and stable in some sense to the theoretical solution y(x), as the step-length, n tends to zero. In Yahaya and Adebgoye (2013) convergence and stability analysis of symmetric implicit Runge-Kutta Method for direct integration of first, second and third ODEs was presented. How to find the order and error constant, test for consistency difference from the traditional R-K process was presented and a plot of RAS was also explained. Stiff ordinary differential equations were not easily handled because of special requirement of A-stability. Agam and Yahaya (2014) presented new three-stage implicit Runge-Kutta type method with error estimation for first order ordinary differential equation based on collocation approach and the continuous coefficient system was evaluated at special Gaussian points. Another novelty of the work is that, for problem without exact solution, this method together with local error can be used to determine its exact solution. Error analysis in R-K Method (RKM) is considerably more difficult than in the case of Linear Multistep Method (LMM) due to loss of linearity in the method. Also the fact that the general RKM makes no mention of the function $f\ (x,y)$, which defines the differential equation, and thus makes it impossible to define the order of the method independently of the differential equation. To address this problem, in 2012 Yahaya and Adegboye presented reformulation of Runge-Kutta Method into linear multistep method for Error and convergence analysis. This idea in the paper enabled authors to determine the order, error constant, Zero stability, consistency and convergence of the RKM independently of the differential equation via a similar process like Linear Multistep Method (LMM).

Mr. Vice-Chancellor Sir, one good thing I must mention about R-K method is the easy application to real-life problems, for example the interaction between neurons and the decarbonisation of lime-stone to produce quick line etc. We considered the mathematical model developed in Odigure *et al* (2009) and Muhammad *et al* (2015 a,b) for the process of limestone. In their study, Odigure *et al* (2009), developed a mathematical model for the process of limestone decarbonisation to produce quicklime (CaO) according to chemical reaction shown in equation

$$CaCO_3 \rightarrow CaO + CO_2 \qquad 5.1$$

It has been reported that the quality of CaO produced is dependent on the chemical and microstructure composition, density and burning conditions (temperature, CO_2 concentration and particle size). The mathematical model for the decomposition of calcium carbonate is represented by the relationship presented in Equation 5. 2 (Odigure *et al*; 2009)

$$\frac{d^2 T_C}{dr^2} + \frac{2dT_C}{r\,dr} - \frac{P_p K_1 C_{c\Delta H}}{K_e} = 0 \qquad 5.2$$

$$\frac{dT_C}{dr} = 0 \qquad at \qquad r = 0$$

Where

K_e = effective thermal conductivities = 3 w/m.k

C_c = concentration of CO_2 in the gas stream

$$\rho_p = \frac{2710kg}{m^3} \quad \Delta H = 5.699\ x\ \frac{10^3 l}{mol} \quad C_c \ = \ 13.204\ molm^3$$

The concentration of the CO_2 in the gas stream can be estimated from the relationship presented in equation (5.2)

$$\frac{d^2 C_C}{dr^2} + \frac{2dC_C}{r\,dr} - \frac{P_p K_1 C_{c\Delta H}}{(DK)_e} = 0$$

$$\frac{dC_C}{dr} = 0 \ \ at\, r = 0$$

The time taken to produce quicklime from calcium carbonate and conversion of calcium carbonate to calcium oxide can be estimated from the relationship shown in equation (5.3) and (5.4) respectively;

$$t = \frac{PA}{3M\ A^{km}\ C_c}\ \frac{(r_s - r)^3}{r^2} \qquad 5.3$$

$$X_A = 1 - \frac{3M\ A^{km}\ C_{c}{}^{r^2 t}}{PA^{r\frac{3}{s}}} \qquad 5.4$$

$$(DK)_e = 1.197\ x\ 10^3 m^2/s$$

$k_1 = Ae\ \frac{E}{R_g T}$ A = 2.01 E = 4.62 x 10^4 $R_g = 8.314 J/mol.K$

$PA = 2710 kg/m^3$ $k_m = \frac{0.3m}{s}$ $M_A = 44kg$

$r = radius\ of\ CaO\ formed$

The performance of the Runge-Kutta type methods on this problem will determine the various temperatures of conversions, the time taken to produce quicklime from calcium carbonate and conversion of calcium carbonate to calcium oxide at various stages of the step numbers (Ks). The temperature of conversion (°C) at various radii (m) is presented in Table 5.1.

Table: 5.1: The various temperatures of conversion obtained at various radii using the proposed K= 1 second order RKTM

r(m)	$T_0 = 600^0\ C$ T	$T_0 = 650^0\ C$ T	$T_0 = 700^0\ C$ T
0.1	634	684	734
0.2	685	735	785
0.3	736	786	836
0.4	787	837	887
0.5	838	888	938
0.6	889	939	989
0.7	940	990	1040
0.8	991	1041	1091
0.9	1042	1092	1142
1.0	1093	1143	1193

Source: *Muhammad, 2016*

The time taken (sec) and rate of conversion (X_A) are presented in Table 5.2:

Table 5.2: The time taken in seconds and the rate of conversion of the product obtained using the proposed K =1 second order RKTM

r_S(m)	$C_0 = 13.204$ C_C	t(sec)	X_A
0.01	13.28916667	0	0.9900000000
0.02	13.41691667	0.005100601362	0.9900000000
0.03	13.54466667	0.04041994985	0.9900000000
0.04	13.67241667	0.1351426977	0.9900000000
0.05	13.80016667	0.3173728321	0.9900000000
0.06	13.92791667	0.6141832359	0.9900000000
0.07	14.05566667	1.051662545	0.9900000000
0.08	14.18341667	1.654959475	0.9900000000
0.09	14.31116667	2.448324770	0.9900000000
0.1	14.43891667	3.455150930	0.9900000000

Source: *Muhammad, 2016*

The rate of conversion $X_A = 0.9900000000$ for the various times, it means that there is 99% conversion of the product.

Yahaya (1989) investigated the problem of vibration of beams subjected to various moving forces/loads and of interest is the effect of damping on the transverse deflection. The analysis was done for the case when the mass of the moving load is small compared to the mass of the Beam. The results show that the amplitude of deflection is smaller when the damping term is retained than when damping term is neglected. The study is of importance to a working engineer who makes use of solution of the Bernoulli-Euler beam without damping. Under the action of any form of moving forces deflection of the beam is exaggerated. It is also useful in the construction of bridges.

5.1 The Multi Step Collocation Method (MCM).

The derivation of LMMs through inter-polation and collocation is now a well known process as evident in Yahaya (2004), Badmus (2012), Adeniran (2013), Agam (2014), Muhammad (2015) and Tijani (2016).

Atkinson (1989) describes collocation as probably now the most important numerical procedure for obtaining continuous methods for ODEs.

The basic method seeks a K-step (K > 0) expansion of the form

$$y(x) = \sum_{r=o}^{p-1} a_r \, Q_r(x) \qquad 5.1.1$$

But $x \in [x_n, x_{n+k}]$

where, a_r are constants and $Q_r(x)$ are

$$Q_r(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{p-1} x^{p-1},$$

and $a_0, a_1, \ldots, a_{p-1}$ are the taylor series constants. This is based on a matrix inverse and an arbitrary basis vector function for the ODEs of the form.

$$Ly = f(x, y), \qquad a \leq x \leq b \qquad 5.1.2$$

And

$$L \equiv \left(\frac{d}{dx}, \frac{d^2}{dx^2}, \ldots, \frac{d^N}{dx^N} \right)$$

Depending on the problem of interest, where y satisfies initial or boundary conditions. The analytical solution y is assumed to exist and unique in [a,b].The expansion (5.1.1) is in terms of a set of known basis functions as defined above, $Q_r(x)$,..r = 0,1,..,p-1, which for flexibility can be chosen as low order piece-wise polynomials, splines, exponential functions, logarithm functions, delay-polynomial functions and rational functions for special application as may be appropriate.

5.2 Global Error Estimator and Convergence

The subject of error estimations is of general concern in Numerical Analysis, since most numerical methods provide approximation to the true desired solution of mathematical problems. It is important to be able to bound or estimate the resulting errors and a numerical method, therefore which fails to provide a suitable procedure for doing this is incomplete.

The spring board for my research on the subject of error analysis of the linear multistep method is the work of Yusuph and Onumanyi (2002). This practical error estimation approach therefore deserves greater attention, see Badmus (2013), Yahaya and Adegboye (2012) Agam and Yahaya (2014) to mention just a few. The construction of a global error estimate to the continuous approximations given in this lecture in earlier sections, is an estimate of the global error involved in the continuous interpolant for the local interval x_n, x_{n+k} for the first order ODEs. Let us recall

$$e_n(x) \equiv y_n(x) - y(x)$$

the error function of the approximant $y_n(x)$ to $y(x)$. The relative error is given by

$$R_n(x) = \frac{e_n(x)}{y_n(x)}.$$

The function $y(x)$ satisfies the differential equation

$$D^r y(x) = f(x, y), \qquad r = 1,2$$

The basic idea is to perform the integration from the step x_n to x_{n+k} with a p^{th} order and $(p+1)^{th}$ order method, to get an estimate of the error in the p^{th} order integration, thereby assess the accuracy of the computed numerical solution.

Example: Global error estimate for the Numerov method
Let us recall; the Numerov equation on $k = 2$ and its hybrid;

where $V = (y_n, y_{n+1}, y_{n+3/2}, f_n, f_{n+1}, f_{n+2})$ and $V = (y_n, y_{n+1}, f_n, f_{n+1}, f_{n+2})$.
Subtracting the later from the former, to obtain:

$$e_n(x) = y_5(x) - y_4(x)$$

$$= \frac{[240(x-x_{n+1})^5 - 800h^2(x-x_{n+1})^3 + 560h^4(x-x_{n+1})]y_n}{375h^5} +$$

$$\frac{[-48(x-x_{n+1})^5 + 160h^2(x-x_{n+1})^3 - 112h^4(x-x_{n+1})]y_{n+1}}{25h^5} +$$

$$\frac{[480(x-x_{n+1})^5 - 1600h^2(x-x_{n+1})^3 + 1120h^4(x-x_{n+1})]y_{n+\frac{3}{2}}}{375h^5} +$$

$$\frac{[-42(x-x_{n+1})^5 + 140h^2(x-x_{n+1})^3 - 98h^4(x-x_{n+1})]f_n}{375h^5} +$$

$$\frac{\left[-126(x-x_{n+1})^5 - 25h(x-x_{n+1})^4 + 420h^2(x-x_{n+1})^3 + 450h^3(x-x_{n+1})^2 - 419h^4(x-x_{n+1})\right]f_{n+1}}{375h^5} +$$

$$\frac{[6(x-x_{n+1})^5 - 20h^2(x-x_{n+1})^3 + 14h^4(x-x_{n+1})]f_{n+2}}{600h^5} +$$

The first attempt on an Error estimation of the LMM was reported in Lambert (1973), where he developed a simple algebraic approach to the first order ODEs as you would see in subsequent sections.

Order and Error Constant
Taylor series method about a point that was used as basis function for the derivation of LMM in previous sections of this lecture will also be in use here.

If we associate the linear difference operator "L" defined by

$$L[y(x);h] = \sum_{j=0}^{k} [(\alpha_j y(x+jh) - h\beta_j y'(x+jh)] \qquad (a)$$

Where $y(x)$ is an arbitrary function, continuously differentiable on [a,b].

The reason for introducing this operator is that, by allowing it to operate on an arbitrary test function $y(x)$, which can be assumed to have as many higher derivatives as we required. On this premise, a formal definition of the order of accuracy of the operator and of the Associated Linear multistep Method, without invoking the solution of the initial value problem (3.1.1 and 3.1.2) which we already observed, may possess only a first derivative.

Expanding the test function $y(x+jh)$ and its derivative $y'(x+jh)$ as taylor Series about x, and collecting terms in (a) gives

$$L[y(x);h] = C_o y(x) + C_1 h y^{(1)} x + \cdots + Cq h^q y^{(q)}(x) + \cdots, \qquad (b)$$

Where the Cq are constants.

Definition: The difference operator (a) and the associated linear multistep method (3.1.1) are said to be of order P, if in (b);

$$C_0 = C_1 = \cdots = C_p = 0; C_{p+1} \neq 0$$

A simple calculation yields the following formula for the constant Cq in terms of the coefficient $\alpha_j; \beta_j$;

$$C_0 = \alpha_0 + \alpha_1 + \alpha_{2+\ldots,} . + \alpha_k$$

$$C_1 = \alpha_1 + 2\alpha_2 + \cdots + k\alpha_k - (\beta_0 + \beta_1 + \beta_2 + \cdots \beta_k)$$

$$C_q = \frac{1}{q!}(\alpha_1 + 2^q \alpha_2 + \cdots + K^q \alpha_k)$$

$$-\frac{1}{(q-1)!}(\beta_1 + 2^{q-1}\beta_2 + \cdots + k^{q-1}\beta_k), \qquad q = 2,3,\ldots$$

These formulae can be used to derive a linear multistep method of given structure and maximal order. See Lambert (1973) for details

Consider the 2nd order ordinary differentials equations

$$Ly(x) = \sum_{r=0}^{k} \mathrm{P}_r(x); \quad y_{(x)}^{(2)} = f(x, y), \qquad a \le x \le b$$

The Linear Multistep Method (LMM)

$$\sum_{j=0}^{k} \propto_j (x) y_{n+j} = h^2 \sum_{j=0}^{k} \beta_j (x) f_{n+j}, \qquad (d)$$

Is said to be of order p, if

$$C_0 = C_1 = \cdots C_p = C_{p+1} = 0; \quad C_{p+2} \neq 0$$

Then $Cp + 2$ is the error constant and

$$C_0 = \propto_0 + \propto_1 + \propto_2 + \cdots + \propto_k$$

$$C_1 = \propto_1 + 2\propto_2 + \cdots + k\propto_k$$

$$C_2 = \frac{1}{2!}(\propto_1 + 2\propto_2 + \cdots + k^2\propto_k) - (\beta_0 + \beta_1 + \cdots \beta_k)$$

$$C_q = \frac{1}{q!}(\propto_1 + 2^q\propto_2 + \cdots + k^q\propto_k) - \frac{1}{(q-2)!}(\beta_1 + 2^{q-2}\beta_2 + \cdots k^{q-2}\beta_k)$$

$q = 2,3\ldots,$

Or

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{21} f''(a) + \cdots + \frac{h^n}{n!} f^n(a) + \cdots \qquad (f)$$

Mr. Vice-Chancellor Sir, much has been said in this lecture on the minimisation of the error of a solution function for the first and second order ODEs. But hardly can you see materials on how to obtain errors for a third and higher order ODEs. Most authors forced the order one and two approaches presented above in this lecture to handle the higher order or better still use the direct Taylor series expansion approach. Thus, justifying the desirability of the new technique. In the year 2010 we derived a formulae implicitly defined by the third order ODES and presented here with an Illustration. See Yahaya and Badmus (2013); Badmus (2010) for other examples and further details.

Proposed Order and Error Constant of Linear Multi-Step Methods for Third Order Differential Equations

With a linear multi-step method

$$\sum_{j=0}^{k} \alpha_j(x) y_{n+j} = h^3 \sum_{j=0}^{k} \beta_j(x) f_{n+j} \qquad (g)$$

We associate the linear difference operator

$$L[y(x), h] = \sum_{j=0}^{k} \alpha_j(x) y_{n+j} - h^3 \sum_{j=0}^{k} \beta_j(x) f_{n+j}$$

Assuming that $y(x)$ is sufficiently differentiable, we can expand equation (g) as a Taylor series about the point x to obtain the expression

$$L[y(x); h] = C_0 y(x) + C_1 h y'(x) + \cdots + C_0 h^q y^q(x) + \cdots \qquad (h)$$

Where the constant q = 4,5…. Are given as

$$C_0 = \sum_{j=0}^{k} \alpha_j$$

$$C_1 = \sum_{j=1}^{k} \mathrm{j}\alpha_j$$

$$C_2 = \sum_{j=1}^{k} \mathrm{j}^2\alpha_j$$

$$C_3 = \frac{1}{3!}\left(\sum_{j=1}^{k} \mathrm{j}^3\alpha_j\right) - \left(\sum_{j=1}^{k} \beta_j\right)$$

$$C_4 = \frac{1}{4!}\ (\sum_{j=1}^{k} j^4 \alpha_j) - (\sum_{j=1}^{k} \beta_j)$$

$$C_5 = \frac{1}{5!}\ (\sum_{j=1}^{k} j^5 \alpha_j) - (\frac{1}{2!}\sum_{j=1}^{k} j^2 \beta_j)$$

$$C_6 = \frac{1}{q!}\ (\sum_{j=1}^{k} j^q \alpha_j) - \left(\frac{1}{(q-3)}\sum_{j=1}^{k} j^{q-3} \beta_j\right)$$

Collecting the corresponding terms together gives the form

$$L[y(x,h] = C_0 y(x) - C_1 h y(x) + \cdots + C_q h^q y^q(x) + \cdots \qquad (i)$$

Where

$$C_0 = \propto_0 + \propto_1 + \propto_2 + \propto_3 + \cdots + k \propto_k$$

$$C_1 = \propto_1 + 2 \propto_2 + 3 \propto_3 + \cdots + k \propto_k$$

$$C_2 = \frac{1}{2!} \propto_1 + 2^2 \propto_2 + 3^2 \propto_3 + \cdots + k^2 \propto_k$$

$$C_3 = \frac{1}{3!}(\propto_1 + 2^3 \propto_2 + 3^3 \propto_3 + \cdots + k^3 \propto_k) - (\beta_0 + \beta_1 + \beta_2 + \cdots + \beta_k)$$

$$C_4 = \frac{1}{4!}(\propto_1 + 2^4 \propto_2 + 3^4 \propto_3 + \cdots + k^4 \propto_k) - (\beta_1 + 2\beta_2 + 3\beta_3 + \cdots + k\beta_k)$$

$$C_5 = \frac{1}{5!}(\propto_1 + 2^5 \propto_2 + 3^5 \propto_3 + \cdots + k^5 \propto_k) - \frac{1}{2!}(\beta_1 + 2^2\beta_2 + 3^2\beta_3 + \cdots + k^2\beta_k)$$

$$C_q = \frac{1}{q!}(\propto_1 + 2^q \propto_2 + 3^q \propto_3 + \cdots + k^q \propto_k) - \frac{1}{(q-3)!}(\beta_1 + 2^{q-3}\beta_2 + 3^{q-3}\beta_3 + \cdots + k^{q-3}\beta_k), \quad ., \qquad q = 3,4 \qquad (j)$$

Following Henrici (1962), we say that the method has order P if $C_0 = C_1 = C_2 = \cdots C_p = C_{p+1} = C_{p+2} = 0,\ \ C_{p+3} \neq 0\ \ C_{p+3}$ is then the error constant and $C_{p+3} h^{p+3} y^{(p+3)}(x_n)$ the principal local truncated error at the point x_n

I want to point out here, that these new formulae could further be used to derive new set of Schemes that could handle effectively

Special Third Order ODEs although one at a time, unlike the continuous method approach. This Construction Process could be termed "***Maximal Order Approach***" as done in Lambert (1973).

CONCLUSION

It is obvious that to assess the history of numerical methods in the 20th century we must first recognize the legacies of the previous century on which it was built. Notable among them includes Adams and Bashforth (1883), Runge (1895), Heun and Kutta-Butcher (1906). The famous Adams-Bashforth method plays an essential role in modern software. Together with the Adams-Moulton method and the practical use of Taylor's series method, following the important and prophetic work of Adams and of Runge, the new century began with further contributions to what is now known in the 21st century as the Runge-Kutta Method by Heun and Kutta in 1900. That is what is now recognized as the starting point for modern one-step methods. See Butcher (2000) for details. Today, Mr. Vice-Chancellor Sir, as you have seen in earlier sections of this lecture, there are many variants of the Runge-Kutta Method in common use, but the most popular is the So-called PECE mode as in Section 4.5 of this lecture. The modern analysis of continuous multi-step methods is intimately bound up with the work of Dahlquist (1956 and 1959) stated in 4.6 of Section 4 of this lecture. This body of works is in several parts, of which the first deals with the concepts of consistency, stability, and convergence discussed extensively in sections 3.0 and part of 4.0. All our works on linear multistep methods show that consistency and stability are together equivalent to convergence. See section 4.0 of this lecture. Development of single step and hybrid block methods that circumvented the Dahlquist stability barrier theorem for the solution of first order initial value problems of ordinary differential equations, and combines the advantages of the block method and allow continuous linear multistep method to address the setbacks of predictor-corrector method were the major contributions I made in my earlier days as a numerical analyst. Although a flurry of activities by other authors followed. I want to state here that the continuous hybrid method combines certain characteristics of CLMMs with

Runge-Kutta Methods with the flexibility of changing step length and evaluating at off-step points. See Section 5.0 of this lecture. As an improvement on the limitation of both the single-step and multistep methods, Yahaya (2004) introduced the concept of CLMM which was earlier briefly discussed in Yusuph and Onumanyi (2002) in their paper titled: New Multiple FDMs through Multistep Collocation for problem 3.1.2 of this lecture. The major feature of the work is that, the resulting piecewise continuous approximate solution (interpolant) belongs to the space $C^1[a,b]$ over subintervals which do not overlap. The success of the above research gave me recognition as I was invited to talk on "Recent Numerical Methods for Solving Differential Equations" in a workshop organised by National Mathematical Centre (NMC), Abuja on the 28th of February 2005. In this lecture Section 5.0 also showed that block methods were formulated in terms of Linear Multistep Methods, but it was able to preserve the traditional advantages of one-step methods of being self-starting and permitting easy change of step-length. Yahaya (2007) also showed that its advantages over Runge-Kutta Methods lie in the fact that they are less expensive in terms of the number of functions evaluation for a given order, the R-K method can also be extended to solve higher order differential equations. The method requires less work with very little cost (when compared with classical R-K).

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Mr. Vice-Chancellor Sir, distinguished ladies and gentlemen, let me say this to this audience and others who may come across this lecture in print: The Prophet of Islam, **Muhammad (SAW)** on the 9th day of Dhul-Hijjah, 10.A.H in the Urana Valley of Mount Arafat delivered his last Sermon. Permit me to quote some part thereof, …."*All those who listen to me shall pass on my words to others and those to others again; and may the last one understand my words better than those who listen to me directly. Be my witness O Allah, that I have conveyed your message to your people."*

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BRIEF PROFILE OF THE INAUGURAL LECTURER

Professor Yusuph Amuda Yahaya was born on the 1st day of March,1966 to the family of Late Mallam Ahmadu Igbeti and Mallama Hanatu Azunmi. He hails from Ilorin, Ilorin-East LGA, of Kwara State. He had his early education at Ansarul Islam Primary School, Oke-male, Ilorin in Ilorin-East LGA, Kwara State from 1973 to 1979. Thereafter, he was admitted to Government Secondary school Malete, Moro LGA, Kwara State in 1979. He completed his secondary education in 1984 and obtained the West African School Certificate Examination (WASCE). Between 1985 and 1989 he went through the B.Sc. (Hons) Mathematics degree at the University of Ilorin. Subsequently he served the nation, through the National Youth Service Corps, (NYSC) scheme between 1989 and 1990, at the Command Secondary School Abakaliki, then in Anambra State. His working Career has been a good mix of Academic and Teaching engagements. He worked briefly as a mathematics teacher at the Okanle-Fajeromi Secondary School in Ifelodun LGA, Kwara State. Before moving to Kaduna Polytechnic in 1991 as Assistant Lecturer. While at Kaduna Polytechnic, he proceeded to University of Jos, for higher degrees. He obtained the degrees of Master of Science (M.Sc.) and Doctor of Philosophy (PhD), both in Mathematics (with specialisation in Numerical Analysis) in 1997 and 2004 respectively. He rose through the ranks to become Senior Lecturer in 2005. A position he held till July 2008 when he transferred his services to Federal University of Technology, Minna as a Senior Lecturer. He became a Professor of Mathematics in October, 2013.

In Federal University of Technology, Minna, Professor Yusuph Amuda Yahaya has held a number of Administrative positions which include: Department of Mathematics Postgraduate Coordinator, July 2008 to 2010; Deputy Dean, School of Science and Science

Education (SSSE), 2009 to 2011; Congregation Representative of SSSE at Senate, 2009 to 2011; Deputy Director, Academic Planning Unit, 2012 to 2015; and Director, Academic Planning Unit, 2015 to 2020. He was elected Chairman, Committee of Directors of Academic Planning of Nigerian Universities (CODAPNU) in 2016 during their Annual Conference at University of Abuja, a position he still occupies till date.

He has published several papers in the areas of Numerical Integrations of both Ordinary and Partial Differential Equations and recently in the area of Delay Differential Equations. He has supervised and still supervising scores of Master of Technology (M.Tech) students. He has also supervised and co-supervised 10 Doctor of Philosophy (PhD) students in FUTMINA.

He is married with Children.

Note