FEDERAL UNIVERSITY OF TECHNOLOGY MINNA

MATHEMATICS, MATHEMATICIANS AND NUMERICAL ANALYSIS: THE BRIDGE AND BRIDGEHEAD VIEW OF NIGERIA WITH MATHEMATICAL PRISM

By

KAYODE RUFUS ADEBOYE, PhD, FMAN, FAC
Professor of Applied Mathematics

INAUGURAL LECTURE SERIES 27

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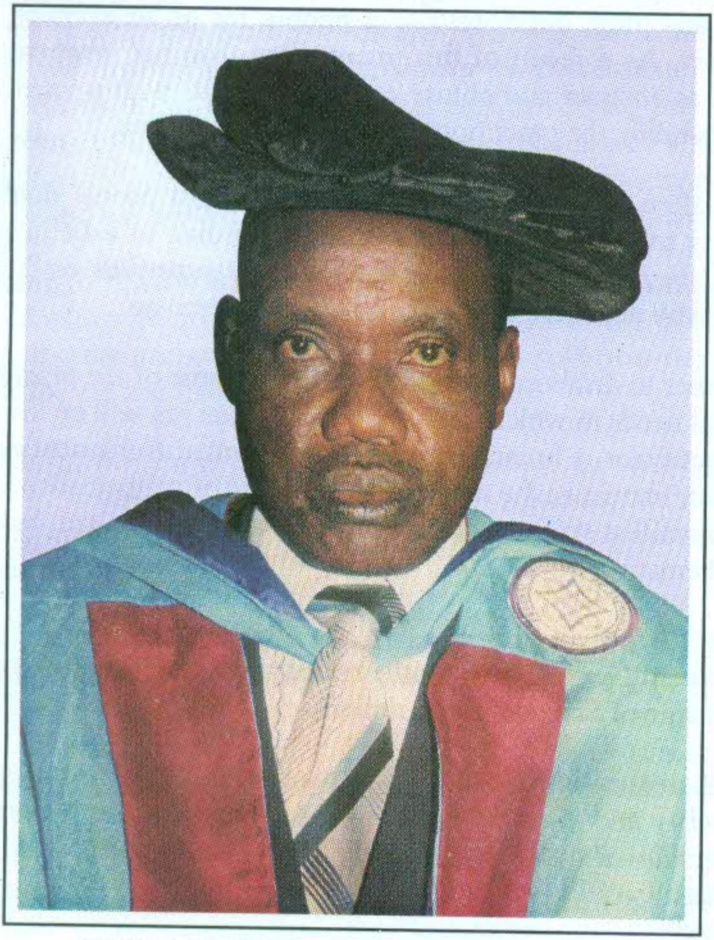

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1.0 Introduction

The motive force of a fixed idea is very great. Many people have this fixed notion or obsession or even complex about mathematics and this often develops into a phobia for or an aversion to the subject. As a result of this, more often than not, mathematics is seen as abstract and obtuse or even esoteric, with a lot of myths surrounding the practitioners – the mathematicians.

Therefore it is not difficult to see why most people doubt their ability to understand mathematics. According to a Latin proverb *"Qui in prdum catamine dubitat tendem superatus est"*. That is "He who doubts in a footrace is always overcome".

In other to disabuse the people of these, some of the practical and social needs to which mathematics has been put will be illustrated. In particular, a branch of mathematics called Numerical Analysis will be shown as the most applicable side of mathematics and will be manifest to all that it actually accentuates the beauty of mathematics.

2.0 Mathematics

According to Kuku, Mathematics is first and foremost the language of Science. Other definitions include:

Mathematics is the study of numbers, shapes, physical systems, patterns and their relationship. Mathematics is the Science of quantitative relations, which has very wide and useful applications.

In actual fact, Mathematics arose from the need for a system for counting and for calculating areas and volumes. Isaac Newton stated *"Numero pondere et Mensura Deus omnia condidit"* (God created things by number, size and weight)

Mathematics is not just about numbers it is also about general objects. It is a process of trying to verify how a statement made by A is related to that made by B. With the use of well-defined notations or symbols, Mathematics can reduce a vast amount of difficult information into a finite number of expressions which when properly understood, help a great deal in solving problems, which relate to the given set of information. The use of letters to represent an unknown quantity has made Mathematics a universal language and has proved to be a magnificent intellectual tool.

According to Francis Bacon, many parts of nature can neither be invented with sufficient subtlety, nor demonstrated with sufficient perspicuity, nor accommodated unto use, with sufficient dexterity without the aid and intervention of mathematics.

Albert Einstein, one of the greatest scientists of the 20th century asked "how can it be that mathematics, a product of human thoughts independent of experience, is so admirably adapted to the object of reality?"

Robert Recorde stated that "Besides the mathematical arts, there is no infallible knowledge, except it be borrowed from them.

And to Leonardo Da Vinci, "No human enquiry can be called true science unless it proceeds through mathematical demonstration."

Ward and Hardgrove [1] defined mathematics by stating that "mathematics results from the discovery, the formulation, the systematic development, and the application of patterns of inductive thinking". In short, they stated that mathematics consists of patterns of related ideas and patterns of thought. And they elaborated by stating that; the development of simple convenient mathematical patterns is an important part of the evolution of the western culture.

Mathematics knows no language, political or ideological barriers to function. That is why Mathematics is even more encompassing than English; the most widely spoken language in the world. The Chinese, Russians and so on, do not speak English yet they practice Mathematics the same way as every other nation does. The complexity of every society, so different, one from another, is responsible for the generation of codes, norms, rules and values in direction of organizing, classifying, comparing and ordering the action of its individuals. Instances of these codes, norms, rules and values are instruments of analysis, of explanations and of action, such as, more or less, small and big, few or many, near and far, and in and out. These codes, norms, rules and values, for instance cardinality and ordinality, counting and comparing, take different forms according to the cultures in which they were generated, or organized and accepted.

According to Galileo "Philosophy is written in this grand book of the universe which continually opens to our gaze, it is written in the language of mathematics". So we can conveniently say that mathematics is the language of the Creator of the universe – God!

The discovery of the Higgs Boson is the latest reminder that the universe can be understood through mathematics. "It makes you feel good as a theorist that mathematics does provide a window of reality" – Brian Greene, Columbia University Physicist's comment on the discovery of the Higgs Boson (4th July 2012).

3.0 Mathematics, Games, Logic and Patterns

Mathematics is logic. It is the discovery and application of patterns in inductive and deductive thinking. It is also a game. According to Bertrand Russell [1], Mathematics is the class of all propositions of the form p implies q, where p and q are propositions containing one or more variables. Therefore, mathematics is simply any assertion of truth or falsehood. In fact all pure mathematics consists of logical deductions. Bertrand believed that mathematics and logic are one and the same thing; so they are inseparable. The development of mathematical ideas actually implies the development of logical thinking.

Mathematics consists of patterns of related ideas and patterns of thought. We observe that these patterns appear in nature. For instance the blossoms of some flowers naturally arrange themselves in interlacing spiral curves. The arrangement of some seeds and fruits like the pineapple and the arrangement of leaves on trees, solar system and the galaxies, all depict specific patterns of arrangement.

In 1772, the astronomers, Johann Titius and Johann Bode [1] discovered that the distances from the sun to the planets, known at the time, showed an interesting numerical pattern. If we take the distance from the sun to Saturn to be 100 (Saturn was the farthest planet known at the time) then the distances of the planets are approximately:

Mercury	4
Venus	$4 + 2^0(3)$
Earth	$4 + 2^1(3)$
Mars	$4 + 2^2(3)$
Asteroid	$4 + 2^3(3)$
Jupiter	$4 + 2^4(3)$
Saturn	$4 + 2^5(3)$

Mercury does not fit the pattern exactly (it should be: $4 + 2^{-1}(3)$ to fit the pattern).

We observe that no planet fits $4 + 2^3(3)$ but Bode insisted that there must be a planet at that distance from the sun. In 1801, Giuseppe Piazzi found, not a planet, but a smaller body called asteroid at the distance of $4 + 2^3(3)$, and before long the celestial police (a group in Germany) found many more asteroids about the distance, showing that these were fragments of a planet that somehow got smashed into pieces. The discovery of the asteroid belt was a great success for Bode's law and this encouraged the search for planets beyond Saturn at distances following the progression: i.e. $4 + 2^6(3)$, $4 + 2^7(3)$, etc. Sure enough, Uranus turned up to be at a distance of $4 + 2^6(3)$! *Quod erat Faciendum* (i.e. that which was to be shown).

Mathematics is not only the language of the sciences; it is the essential nutrient for thought, logic, reasoning and therefore, progress, technology and indeed all human progress. Mathematics liberates the mind and not only gives individuals an assessment of their intellectual abilities, but points to the direction for improvement. Prof. Jibrin Aminu [1] said that "Mathematics is the gift of God to man and it is only those who are devoted to extolling the supremacy of God, who is the only Being that can explain to us the phenomena of infinity and the incongruity of the square root of -1". Of course according to Pascal; *"Tout ce qui est incomprehensible ne laisse pas, detre, le numbrebinfiniti un espace infini cealaufini"* [1] (What is incomprehensible exists; infinite number or infinite space equal to a finite one!).

4.0 Branches of Mathematics

Mathematics consists of at least two distinct subjects [2]: some call them Pure and Applied Mathematics but some refer to them as Mathology (Pure maths) and Mathophysics (Applied maths) respectively.

For the professional pure mathematician, mathematics is the logical dovetailing of a carefully selected sparse set of assumptions with their surprising conclusions via a conceptually elegant proof. Simplicity, intricacy and above all logical analysis are the hallmark of mathematics.

5.0 Applied Mathematics

When the procedures are turned to the analysis of physical and engineering problems and when the relations are examined by the methods developed in pure mathematics, the field is called Applied Mathematics or Mathematical Physics. The general trend now is that every branch of science and its mode of explanations depend more and more on mathematical formulations. There is no doubt that the fountainhead, the inspiration of mathematics is the physical and social universe. The ultimate goal of applied mathematics is "action" from the theory of convex sets, convexity has become the main tool in Linear programming, an indispensable part of modern economic and industrial practice. Functional analysis has become the main tool in quantum theory and particle physics. The Physicist regards applicability of Von Neumann algebra (a part of functional analysis) to elementary particles as the only justification of quantum theory.

Applied mathematics is not engineering; the applied Mathematician does not design airplanes or atomic bombs. Rather, it is a part of theoretical science concerned with the general principles behind what makes planes fly, bombs to explode and bridges not to collapse under load pressure. However the relationship between applied mathematics on one hand, physics and engineering on the other is symbiotic, in the sense that neither can survive without the other. The only way of comprehending the physical universe is by a direct use of creative mathematical imagination.

According to quantum mechanics, the states of any physical object form a linear space. Before quantum mechanics was invented, classical physics was always non linear and linear models were only approximately valid. After the invention of quantum mechanics, nature itself was found to be precisely linear, and the theory of the linear representations of Lie algebra was the natural language of particle physics (Lie algebra was developed by Sophus Lie).

The Schrödinger equation describes correctly everything we know about the behavior of atoms. It is the basis of all of chemistry and most of physics. So the square root of minus one which Schrödinger added to the heat conduction equation helps to show that nature also works with complex numbers. Up till that time, people looked at complex numbers as a set of artificial construction invented by mathematicians to aggrandize themselves! [3]

To be a scholar of mathematics, one must have talent, insight, creativity, imagination, ingenuity, concentration, and drive. Mathematics is not a mob science. It is not a team science. A theorem is not a pyramid; inspiration has never been known to descend on a committee or group. Although most of mathematical creation is done by one man at a desk, at a blackboard, or taking a walk or sometimes by two men in conversation, mathematics is nevertheless a sociable science [5].

Mathematics (this may surprise or shock you) is never deductive in its creation. The mathematician at work, makes vague guesses, visualizes broad generalizations and jumps to unwarranted conclusions. He arranges and re-arranges his ideas and often with the question *"Quod Sectabor Iter"* (what road shall I follow) until he becomes convinced of their truth long before he can write down a logical proof. Einstein had enormous respect for the power of mathematics to describe the workings of nature and an instinct for mathematical beauty which led him unto the right track to find nature's laws.

The Bible says:
"In the beginning was the Word and the Word was with God and the Word was God … All things were made by him and without him was not anything made that was made" (John 1, verses 1-3). Then by analogy, we can comfortably say: "In the beginning was mathematics and mathematics was with God. All things were made with it and without it was not anything made that was made". When succinctly put, we can say – Mathematics abounds in everything made by God!

6.0 The Mathematician

If one asks a mathematician to explain what he/she works on, one will usually be met with rather, a sheepish grin and told that it is impossible to do so in a short time. If one further asks if this mysterious complicated work has practical applications (and we all get asked this question from time to time), then there are typical responses, none of which is immediately impressive. Like G. H. Hardy's case, "the main criterion of mathematical worth is beauty"! However, there are mathematicians who work in areas of acknowledged practical importance such as engineering mathematics, financial mathematics, numerical analysis, theoretical computer science and statistics. To be candid, most mathematicians would be delighted if they proved a theorem that is useful outside mathematics; though not intentionally. It turns out that a great deal of the research in even the so-called practical areas is in fact not practical at all. This is due to the fact that, rather than studying the world directly, mathematicians create 'models' of it and study them. If one works in a practical area of mathematics, then there would be two conflicting criteria for what makes a good model: on the one hand, the model should be accurate enough to be useful and on the other hand, it should be simple and elegant enough to generate realistic and interesting mathematical problems. So the mathematician does benefit society and occasionally produces breakthroughs of enormous economic benefit either directly as in the case of John Nash's equilibrium and public-key cryptography or indirectly by providing the necessary theoretical underpinning to science and technology [4].

The mathematician can be a shy and even occasionally stand-offish person but behind that severe mien is an affable *bon viveur*, effervescent and witty personality often with mordant wits.

The mathematician can be anything but, he/she is <u>not</u> that always frowning, wicked, brash, abrasive, emotionless, a prime donna egoistic person, that puts his/her nose in the air and does complicated mathematics for self-aggrandizement. The quintessential mathematician is essentially an eager beaver, who keeps his/her nose to the grindstone, a problem solver and takes delight in solving problems, though may choose to be unusually or unpredictably rigorous in his approach, making the simple mind to lose track of events with a hollow appreciation or joy. It is like this saying in Latin: *"Flectere si nequeo Superos acheronta movebo"* (If I cannot bend the higher powers, I will move the infernal regions). Many a times, mathematicians, with their idiosyncrasies and largely unfathomable but eccentric minds, often engage in some oddly intriguing manipulation of symbols, though according to strict and ordered processes or techniques bordering on trial and error, to achieve mathematical breakthroughs. All these notwithstanding, however, there are no gainsaying the fact that mathematicians have very discerning, analytical and liberal minds. They are neither parochial nor egocentric. Of course, openness, academic freedom and liberal attitude are the sinews of the university system itself.

According to Prof. Jibril Aminu in [1], he wrote and we quote "Mathematicians are a rare breed of people who have devoted their lives, energy and intellectual capabilities to the pursuit of science, to the pursuit of the abstract, to the immaculate gymnastics which tune up the human brain, escalate human achievement and increase the gap between human beings and other members of the animal kingdom".

7.0 Science / Technology

Science as defined in Webster Dictionary – the state or fact of knowledge; and systematized knowledge derived from observation, study and experimentation carried out in order to determine the nature or principles of what is being studied.

Technologies are methods and processes used by people to enable them provide their various needs as food, shelter, clothing, transportation, communication, good health and security. Science is knowledge obtained by observations.

A good educational system therefore creates a sound base for scientific and technological development. It encourages people to make use of both their brains and hands to acquire suitable knowledge and skills to design and produce methods and processes that will enable them make maximum use of their resources for the benefit of the society.

Technology ranges from the predominantly manual processes of the rural communities of the under-developed to the highly-mechanized and computerized processes of the urban conurbations of the advanced countries [6]. Technology involves many disciplines and expertise as well as precision. In order words, accurate measurement of all sorts – weight, capacity, length, distance, time, rate of change, etc. are important. Infact, Johann Kepler (1610), a renowned mathematician said "perfect knowledge is always mathematical".

Most people, even in the developed countries, will not be enamoured by science. On the other hand; everyone confronts and is affected by Technology in the pursuit of his/her daily activities. So we are only crazy about the wonders of technology. Science is important because it is the application of science to technology that has made it possible for technology to develop so fast in the last 150 years. It is therefore not realistic for a society to love the fruits of technology and at the same time ignore or loathe the study of science. In actual fact, science as a pursuit has always belonged to a small but aristocratic group of people in the society. According to Lyn White [6], "Nature science conceived at the effort to understand the nature of things, had flourished in several eras and among several peoples. Similarly there had been an age-old accumulation of technological skills, sometimes growing rapidly, sometimes slowly. But it was not until about four generations ago that Western Europe and North America arranged a marriage between science and technology. Science was traditionally aristocratic, speculative, and intellectual in intent; technology was lower class, empirical, and action-oriented."

Among the physical sciences, physics is the one that is most served by mathematics. In order to be able to describe the motions in a plane or space the physicists must have a good understanding of vector algebra. More advanced mathematics is needed in the treatment of collision problems and the general theory of coupled oscillations is best treated by means of linear transformation in vector spaces. In the treatment of vibrating strings and of the motion of fluids, emphasis is on the fundamental concepts and mathematical methods used in the mechanics of continuous media. Lagrange's equations are the fundamentals of advanced dynamics .Hamilton's equations and the concept of phase space are prerequisites to courses in quantum mechanics or statistical mechanics. Mention any section of physics, optics, electricity or thermodynamics, mathematics forms the basis of understanding it and also of its application. At higher level, a good knowledge of calculus is essential in understanding problems in chemical thermodynamics, Chemical kinetics and the laws of mass Action. In kinetics theory of gases, the distribution of molecular speeds is accounted for by use of the Maxwell-Boltzmann distribution function of the form:

$$dN/_{N=4}\pi\left\{\left[\frac{m}{2KT}\right\}\right]^{\frac{3}{2}} \exp\left\{-\frac{mv^2}{2KT}\right\} dv \quad [5]$$

where m represents the molecular mass, k the Boltzmann's constant ,T the absolute temperature, while exp is equal to 2.71 being the base of natural logarithm ,v is the molecular speed, *N* has speeds between each of values v and v + dv. In understanding the structure of crystals and molecules, knowledge of symmetry in mathematics provides a useful insight.

Engineering is the application of science to technology. It is customary these days to turn to science to guide us as to the general laws of nature to apply and this has always been the case since technology existed long before science, the former emanated from man's survival instinct while the latter was nurtured by his drive for self-esteem. In fact, it is technology that makes science possible since man had to survive before he could begin to appreciate nature and begin to study its behavior. The evidence before us is that science and technology initially developed along separate paths. The field of technology was controlled by a guild of artisans who regulated the entry and practices within the profession. Under this arrangement, craftsmanship was acquired through tutelage under a master who gradually trained the apprentice over a long period in the arcane mysteries of the trade. Such trades which initially developed around agricultural implements, building and construction, flourished to give the practitioners considerable status and power over construction standards. The engineer's position within the technological order has been consolidated with the increasing interaction between science on one hand and the technical arts on the order [6].

When viewed from the point of technology, we have already noted that science is the chief reason for progress of technology in the past few centuries. Achievements in technology grew exponentially as people learned to see the connection between science and technology. This application is chiefly all about engineering so it is not possible to become a good engineer in the modern world without first being a scientist. In actual fact, the training of engineers is mostly hinged on a thorough grounding in physics and mathematics. To that extent, mathematics is not only the language of physics but all of the sciences. So mathematics is a sine-qua-non for engineering and hence modern technology. Thus, to be a good engineer, one must not only be strong in mathematics but must also know how to translate theoretical, scientific concepts to practical entities and vice-versa. In order words, one must be good in modeling and numerical analysis.

Hence the vital bridge we are talking about! A bridge is a structure of wood, stone, brickwork, steel, concrete etc. providing a way across a river, canal, railway etc. It is a critical factor in social and economic development and military operations or warfare. Now, UNTO THE BRIDGE!

8.0 Numerical Analysis

It has been realized since Newton that a large number of physical phenomena can be described by means of what we know as partial differential equations e.g. the behavior of water waves, the flow of heat, the motion of fluids, the growth of population, spread of disease, etc. Many of these equations are non-linear and cannot easily be solved or may not be solvable analytically. The only way such equations can be handled is by numerical methods. Since many of the equations, though unsolvable by mathematical manipulations, describe real physical problems in engineering and technology, we just cannot ignore them. We just must find a way of solving and analyzing them. That is the essence of numerical analysis.

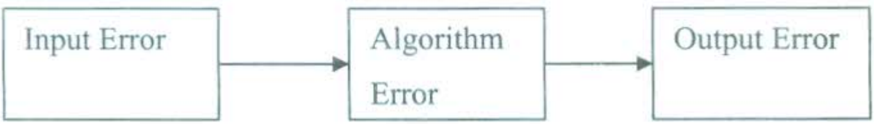
John Von Newman was one of the mathematicians who participated in the Manhattan Project to develop the first atomic bomb. His research in this area led to a belief that the key lay in the development of Numerical Techniques to solve equations, as opposed to using the classical methods of calculus, which generally fail for the kinds of problem which arise in real life.

Numerical Analysis aims at providing practical procedures for calculating the solution of problems in applied mathematics to a specified degree of accuracy. Numerical Analysis involves the development, construction, derivation, description, analysis and evaluation of methods for computing required numerical results for given data. It is part of information processing. Computer and indeed computer science developed from Numerical Analysis.

The given data are the input information, the required results are the output information and the method of computation is known as the algorithm.

The Presence of Errors

Speed and accuracy are very important in the choice of an algorithm. Since input information can rarely be exact, because it comes from measurement devices there must be input error. So we have input error, algorithm error and output error.



8.1 Finite Element Procedures

As finite element analysis spreads to designers who may not have formal training in or good knowledge of numerical analysis, practitioners must ask whether the most appropriate techniques are being used and also whether these techniques are producing accurate details.

We observe that Finite Element Methods (FEM) are widely used in today's engineering practice, through various general purpose computer programs. These techniques are, to an increasing extent being used to make good new designs and improve designs with respect to performance and cost. Considering the crucial role that FEMs play in various areas of engineering and physics, practitioners must be sure of the appropriate methods to use. We take a look at its historical background.

8.2 The Historical Background

The finite element method has come to be closely associated with the use of piecewise continuous polynomial functions defined over each element. This idea dated as far back as 1943, owed its origin to Courant [7], when he used piecewise continuous functions defined over triangular domains and the principle of minimum potential energy to study the St. Venant torsion problem.

However, much of the credit for the development of the finite element method should , by right, go to the engineers who, with tremendous effort using both empirical and ingenious approaches, had developed the method naturally for problems in structural analysis from the principle of virtual work in the early 1950s, see Turner Clough, et al [8].

In 1954 Argyris [13], derived the energy theorems from the principle of virtual work or virtual displacement and the principle of complementary virtual work, which were the basis of the strain-energy approach to structural analysis. Argyris also used the Rayleigh-Ritz approach with the approximate method of displacement analysis for three-dimensional problems in aircraft engineering. This formed the basis for the derivations of the finite element method. The fact that the finite element method greatly resembles the Rayleigh-Ritz method makes it easy to be put into variational form and this formed the basis for the mathematical development and analysis of the method.

The first error estimates for the finite element approximations could be traced to Synge in 1957. Although he did not use the term "finite element" in his book, "The Hypercircle in Mathematical physics" he demonstrated how polyhedral functions (pyramid functions) based on suitable triangulation could be used to approximate by linear interpolation, the function and its first derivatives. He obtained error estimate of order two for the function and of order one for its derivatives. This method of approximation which we have traced back to Courant came to be known as the "finite element method" a term which was first used by Clough [17].

This development in the engineering world aroused the interest of many mathematicians in the finite element method and they started exploring the use of piecewise continuous functions for the approximation of elliptic boundary value problems with smooth solutions, and error estimates were obtained. See Ciaret, et al [9], Pian and Tong [10] and Rvachev and Shklyarov [11]. Within a few years more specifically, between 1968 and 1971, various error estimates for the finite element approximations were obtained by different authors using different variational principles such as Ritz – Galerkin, least squares and collocation methods and for various types of trial function spaces. It was therefore very difficult to say who originated error estimates often associated with the finite element method. However, Strang [12], approximated the solution of an elliptic boundary value problem of order 2m by the finite element method in 1971 and derived the upper bound for the error estimate of order 0(h 2k-2m) in the energy (k is the order of the approximating polynomial).

Furthermore, in 1973, Strang and Fix [14], made use of what was termed "Nitsche's trick" [15], to develop a systematic approach to the derivation of convergence and error estimates for the finite element method. In particular, the Nitsche's trick made it possible to determine the accuracy of the approach in a lower order norm than the energy norm which was hitherto used.

Initially, the finite element method was developed for solving problems in structural analysis. But in 1965, Zienkiewicz and Cheung [16] stated that the method could be applied to all field problems, e.g. heat condition, seepage flow e.t.c.

These efforts led to the development of isoparametric elements by Ergatoudius, Irons and Zienkiewicz [18]. They used isoparametric elements which made it easier than before to approximate problems with curved boundaries. In particular, an isoparametric transformation has curved co-ordinates and this helped to approximate the shape of the boundary of region, under investigation, provided the polynomial or spline function used also approximated the boundary.

Strang and Fix [14], noted that the displacement error alternated in signs and that it was most likely that the approximate displacement u^h of u, would be of exceptional accuracy at the points where these changes in signs occurred. This spectacular observation was also made by Barlow [19] who pointed out that there existed points where exceptional accurate stresses occurred (stresses being obtained by differentiating the appropriate solution for the displacement). These points of exceptional accuracy were later identified to be Gauss points.

8.3 Super - Convergence

This idea of obtaining greater order of accuracy at the knots for the functions and its derivatives than globally (i.e. in the region as a whole) was made clearer by Douglas and Dupout in 1972 in the paper "some Superconvergence results for Galerkin Solution of the two-point Boundary value problems" and hence the beginning of the use of approaches for different types of problems to obtain superconvergence results [20].

In 1976, Dupout, came out with the unified theory of superconvergence for Galerkin methods for two-point boundary value problems which incorporated all the methods and results earlier achieved. He made use of some auxiliary computation with Green's function and obtained superconvergence results for both the function as well as its derivatives at all points in the interval [21].

The relative difficulty in satisfying the continuity requirements especially in problems requiring C^1 or C^2 which occurred in plates and shells influenced the decision of some engineers' who had to consider ways of circumventing or weakening the continuity requirement without at the same time reducing the order of accuracy. What they obtained by this procedure was optimal and this was "a case of apparently obtaining something for nothing". They used the method of least squares on isoparametric elements which actually involve numerical integration. They had noted that low order Gauss integration advantageously involved less computation because fewer Gauss points than necessary were used and moreover, the degree of the approximating polynomial was less. The technique was called Reduced Integration [22]. This would actually qualify as the engineer's own equivalence of superconvergence. There had been and there is still a lot of apprehension about it because it violated the continuity requirements for such problems.

9.0 Research Work

9.1 The Super-Convergence Error Estimate

We consider the equations of the form:

$$LU = f; \text{ in } \Omega \tag{1}$$
and $D^{\alpha}U = 0, on\ \partial\Omega, |\alpha| \leq 2m - 1.$

Where Ω is a bounded domain and L is a linear elliptic operator of order 2m, with $m \geq 1$, an integer.

This is the general form of boundary value problem in engineering and physics e.g. problems of structural mechanics such as deformation, trusses, stress analysis of automotive, aircraft, building and bridge structures, magnetic flux and seepage; soil mechanics, heat transfer, fluid flow, etc.

If $f \in H^{s}(\Omega)$, then, $U \in H^{2m-s}(\Omega)$ which is the regularity hypothesis on L.

A formal integration of (LU, V) m times by parts gives the bilinear form

$$B(U, V) = (f, V) \tag{2}$$

B(U, V) defines an inner product for the energy space and $\sqrt{B(U,V)} \geq 0$, is the associated energy norm. Then $\exists$ a positive constant C_1 such that

$$B(U,V) \leq C_1 \|U\|_m \|V\|_m$$

If U^h is the Galerkin approximation then

$$B(U^h, V) = (f, V) \tag{3}$$

Subtracting (7) from (6), we obtained

$$B(U - U^h, V) = 0$$

which shows that the Galerkin approximation is orthogonal to V, w. r. t. the energy inner product B(U,V).

It can be shown that

$$\|U - U^h\|_m \leq C_2 \|U\|_k h^{k-m} \tag{4}$$

where k is the order of the polynomial in the solution space.

Convergence of the finite element method follows if $k > m$. There is also a connection between the Galerkin approximation and the interpolation theory.

From the notation in the figure (1) below, we have

$$B(U-U^h) \leq B(U-U_I, U-U_I) \tag{5}$$

Since U_1 is the q-interpolant of U, we can easily estimate the R.H.S. of (5) thus:

Suppose we interpolate U by $U_1 = P_k(x)$ a picewise polynomial of order k (i.e. of degree (k-1), then the error due to interpolation is:

$$\amalg U - U_I \amalg_0 \leq C_5 \amalg U^{(k)} \amalg_0 h^k \tag{6}$$

From this, we can deduce that

$$\amalg U - U_I \amalg_m \leq C_6 \amalg U^{(k)} \amalg_k h^{(k-m'')} \tag{7}$$

The error estimate (7) can be expressed in weaker norms than $H^s(\Omega)$ by using a method by Nitsche [22] which is often referred to as Nitsche's trick

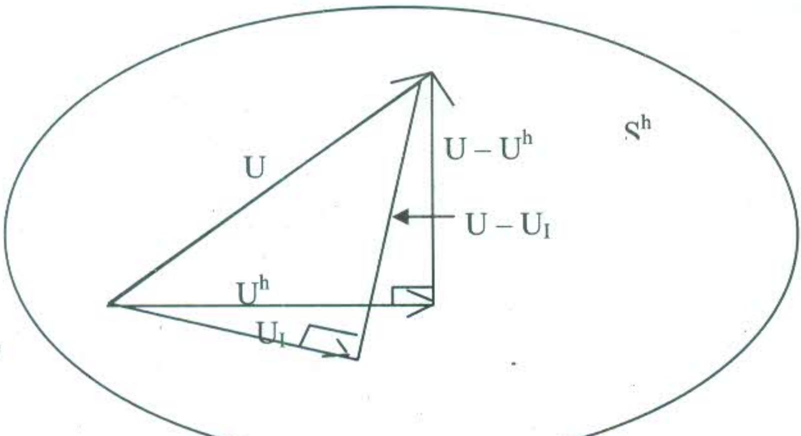


Fig. 1

The general form of Nitsche's trick requires

$$\amalg . \amalg_{-s}, 0 \leq s \leq m'$$

By definition:

$$\amalg \cup \amalg_{-s} = \max_{V \in \partial\ell^{s}} \frac{\left|\int UV\right|}{\amalg \vee \amalg_{S}}$$

Or equivalently:

$$\amalg \cup \amalg_{s} = \max_{V \in \partial\ell^{-s}} \frac{\left|\int UV\right|}{\amalg \vee \amalg_{-S}} \tag{8}$$

By some manipulation of Nitche's trick, the error term (7) increases to

$$\left\|U - U^{h}\right\|_{m} \leq \left\|U\right\|_{k} h^{2k-2m} \tag{9}$$

which is of order two times that of the normal error term (3) and this we refer to as super-convergence, a special phenomenon which obtains at stress points.

In essence, we are saying that it is possible to achieve this extraordinary high order of accuracy in a numerical method if the method is skillfully developed and this characterized all the numerical methods we designed. These numerical methods not only give high order of accuracy for the function but also for all its derivatives, which is a strange phenomenon but novel idea in numerical analysis.

9.2 Super-Convergence Results for Galerkin Method for Parabolic Problems

We first transform the parabolic problem to a two-point boundary value problem. Then, we define a Galerkin approximation procedure for the two-point boundary value problem and finally use the results of Dupont and Wheeler to arrive at superconvergence results for the space variable.

In a nut shell, we consider the parabolic initial value problem:

$$c(x)u_t(x,t) = -Lu(x,t), \quad x \in (0,1), \quad 0 < t \leq T$$

$$u(x,0) = u_0(x), \quad 0 \leq x \leq 1$$

and

$$u(0,t) = u(1,t), = 0 \quad 0 \leq t \leq T$$

The Laplace transform is:

$$sc\hat{v} + L\hat{v} = -\frac{L\hat{u}_0}{s} = \frac{\hat{f}_0}{s}$$

with $\hat{v}(0,s) = \hat{v}(1,s) = 0$

The existence of the inverse Laplace operator $(L + sc)^{-1}$ and the uniqueness of the solution of the problem for any s in the trapezium ABCD (see fig. (2) below) have been proved by Cerutti and Parter, where;

$$s \in \{ s | Re(s) > \lambda_0 \}$$

with λ given by $Lv = \lambda cv$

since $\frac{\hat{f}_0}{s}$ has a simple pole at s = 0 and we observe that $\hat{v}$ and $\hat{v}^h$ are analytic except possibly at $s = 0$, we need only consider the residue at $s = 0$

The Galerkin method is defined by

$$L(\hat{v}-\hat{v}^h)+sc(\hat{v}-\hat{v}^h)=\frac{1}{s}(\hat{f}_0-\hat{f}_0^y)$$

Finally, we summarize the result with the following Theorem:
Let s_l and r be integers such that $3\leq s_1\leq 2r$. Then for h sufficiently small,

$$\left|(u^h-u)\right|+\left|(u^{h'}-u')\right|\leq C\sum_{i=1}^{N}\|u\|_{w1}\, s_{1(l_i)}\, h_i^{s1}$$

which is the super convergent error estimate we wish to show.

Quad erat Faciendum (QEF) (see Ref 45-66)

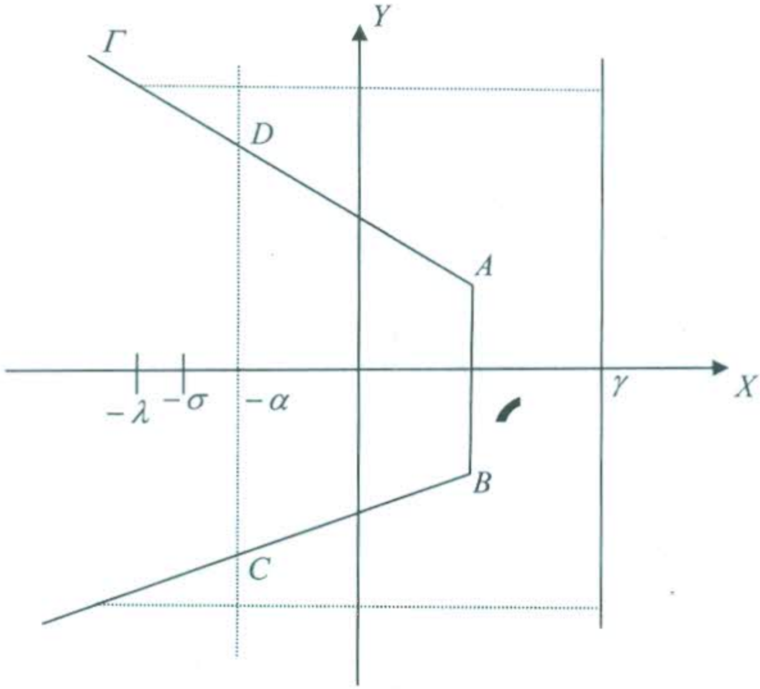


Fig. 2

9.3 C^0 – Collocation-Finite Element Method for Elliptic Partial Differential Equations

A C^0 – collocation-finite element method is defined for approximating the solutions of elliptic partial differential equations. We make use of the results of Douglas, Dupont and Wheeler [26] to obtain super-convergence. Up till then, no one has used this procedure for this kind of problem.

We consider the problem:

$$Lu = -\nabla(a\nabla u) + cu = f, in\ \Omega$$

and

$$u(x_1, x_2) = 0,\ on\ \partial\Omega$$

where

$$Lv(x_1, x_2) = \frac{\partial}{\partial x_1}(x_1, x_2)\frac{\partial}{\partial x_1}v(x_1, x_2)$$
$$-\frac{\partial}{\partial x_2}(a_2(x_1, x_2)\frac{\partial}{\partial x_2}v(x_1, x_2)) + c(x_1, x_2)v(x_1, x_2)$$

and

$\Omega = I \times I$, where $I = (0,1)$.

We consider basis functions of the form:

$$J_{r-1} = \frac{1}{(2r)(2r-1)...(r+2)}(x_l(x_l-1)^{-1}\frac{d^{r-1}}{dx_l^{r-1}}(x_l^r(x_l-1)^r) \qquad l = 1,2,$$

which is the Jacobian orthogonal polynomial. In finite element methods, the use of orthogonal polynomials as basis functions often yields superconvergence.

Then the C^0 – collocation – finite element method is defined as

$$(Lu)(\xi_{ij},\tau_{kn})= f(\varepsilon_{ij},\tau_{kn}),\ i,k=1,2,...N.$$
$$n,j=1,2,...r-1$$

and

$$B(u,v)=(f,v), \forall v\in M_0^0(1,\Delta)$$

where

$B(.,.)$ is the bilinear form: $B(\phi,\psi)=(a\nabla\phi,\nabla\psi)+(c\phi,\psi)$

Thus, the discrete bilinear form is defined by:

$$<\phi,\psi>_{ik}=\sum_{n,j=1}^{N} \frac{A_k A_i\, \phi(\xi_{i,j},\tau_{kn})\psi_2(\xi_{ij},\tau_{kn})}{(1-\xi_j)^2(\xi_j)^2} + (\phi,\psi_1)_{ik}$$

and

$$(\phi,\psi)_{ik} = \int_{I_{ik}} \phi\psi dx_1 dx_2$$

The C^0-collocation-finite element procedure defined above was used in conjunction with the tensor product method.

So we have,

$$<a\nabla^2\xi,V>=\sum_{k,i=1}^{N}-a(x_{ik})(\nabla\xi,\nabla\nabla)_{ik}+\sum_{k,i=1}^{N}<q_{ik}\nabla^2\theta\xi V>_{ik.} \qquad \forall V M_0^{-0}(r,V)$$

Where, by the definition, we have

$$<q_{ik}\nabla^2\xi,V>_{ik} = \sum_{n,j=1}^{r-1} q_{ik} \frac{A_k A_i h_{ik}\nabla^2\xi(\xi_{ij},\tau_{kn})V(\xi_{ij},\tau_{kn})}{1-\xi_j)^2\xi_j^2}$$

Using the H^1 – projection method of Douglas, Dupont and Wheeler, we have the theorem:

Suppose that $a\in C^2(\overline{\Omega}),\ t_0\geq 1\ and\ k'=0$. There exist a constant C depending on τ_0 and $\|a\|_{W^{\infty,2}}$ such that, if u and U satisfy equation above, and $\tau\leq\tau_0$, then,

$$\|U-u\|_{L^\infty(\Omega)} \leq C\left(\|u\|_{r+2} + \|u\|_{W^{\infty,r+1}}\right)h^{r+1}$$

where

$$\|V\|_{W^{p,s}} = \sum_{|\alpha|\le s} \|D^{\alpha}V\|_{L^{P}(\Omega)}, \; V \in W^{p,s}$$

and finally the theorem:

Suppose that $k' = 0, a = 1, \tau_0 \ge 1$, then there is a constant C such that for $t \le \tau_0$,

$$\max_{ij} \left|(U-u)(x_{ij}, x_{2j})\right| \le Ch^{r+2}(x_{ij}, x_{2j})\|u\|_{r+3}$$

The error estimate is the same as that obtained by Douglas, Dupont and Wheeler, but different from that obtained by Wheeler. Wheeler obtained superconvergence results at the knots in one-dimensional problem by using the auxiliary computation constructed by Dupont [40], without the use of the auxiliary computation, the knot estimates will be optimal. Here we used the 'Tensor product method' of Douglas, Dupont and Wheeler [26] which gave a superconvergent result at the knots for the case with C^0-continuity conditions. This is well suited to the method used above, which is itself, an extension of the C^0-Finite element method

9.4 Mathematical Model

We know that FEM is used to solve a mathematical model resulting from an idealization of the actual physical problem under consideration. Normally, the mathematical model is based on assumptions made regarding the geometry, material conditions, loading and displacement boundary conditions. Usually the governing equations of the mathematical model are in general, partial differential equations subjected to some boundary conditions. In most cases, these equations cannot be solved in closed analytical form. Therefore, engineers and practitioners in general, resort to the FEM to obtain numerical solutions. For example, let us consider the analysis of a valve housing axisymmetric geometry and axisymmetric loading. In such a case, it is reasonable to assume axisymmetric conditions for analysis. The complete mathematical model and thus the analysis problem is obtained by specifying the geometry and dimensions, support conditions, material constants and loading.

While engineers cannot, in general obtain analytically the exact solution of the mathematical model in closed form, even when it exists and is unique, an approximate solution can be obtained with very high accuracy, using the FEMs. Hence, the notion of convergence comes in. For example, if E denotes the exact (usually, unknown) strain energy of the mathematical model and E_h is the corresponding strain energy obtained by finite element method, then convergence means

$$\lim_{h \to 0} \left| E - E_h \right| = 0$$

(h is the mesh size of the element).

As the element size h is decreased, the approximate strain energy $E_h$approaches the exact value, E. The more accurate the method, the faster the convergence. Obviously, higher-order elements will reduce the error rate with mesh refinement than lower-order elements.

9.5 **Reliability**

In FEM, reliability means that in the solution of a well-posed mathematical model, the FEM procedures will have two attributes:

(i) Convergence: as earlier explained

(ii) Robustness: The quality of the finite-element solution does not change drastically when the material data (or thickness of a shell) changes.

These attributes are crucial.

If (i) is violated then with mesh refinement, a solution which is not exact is approached and thus could lead to a wrong or faulty design decisions and disastrous consequences. (of course FEM that violates (i) should not be used). Now, for condition (ii) i.e. robustness. Assume for example, that the valve housing is made of steel (Young's Modulus is 200,000 megapascals, Poisson's ratio, is 0.30). The analysis using a reasonable mesh has given acceptable results (That is, the error $|E - E_h|$, is acceptably small). Suppose that we now change the material to a plastic whose Poisson's ratio is 0.50. This change in material condition might result in a relatively small change in the exact solution and it should then result only in the corresponding small change in the FEM. That is saying that the method relatively is immune to input error. However, unfortunately, the FEM that violate (ii) and yield solutions grossly in error when there is a little change in the data, are still being used [23].

9.6 **Adeboye Corrector-Predictor Method [24]**

Throughout the world over, the problem of a mathematician talking to an audience like this which consists largely of non-mathematicians is enormous. While he would want to do justice to the subject, he at the same time would not want to inundate it with mathematics. Perhaps he may have to abide by the Occam's Razor principle [1] which says *"Entia non Sunt multiplicando praeter necessitatem"* (Assumption to explain a phenomenon must not be complicated beyond necessity).

Solution of equations of the form; $f(x) = 0$, by numerical methods.

The simplest but most popular numerical method for the solution of such equations is the Newton-Raphson's iterative method:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad n = 0,1,2,\ldots$$

Though, by numerical standard, the accuracy is very low it is very useful, versatile and widely used both by scientists and engineers. We say such methods, in modeling are very fruitful because they have generality and are also inspiring. These made me take a second look at the method and see how a more accurate method could be evolved. We came up with a new and more accurate method titled "A Predictor-Corrector method for the solution of Algebraic Equations" on a Christmas Eve!

The method is now generally called Adeboye's Corrector-Predictor method. I can say this is one of the simplest things we did but it is the most popular. Just to confirm that "Simplicity is the ultimate Sophistication."

(i) Adeboye's predictor-Corrector Method for the solution of algebraic Equations. We consider the equations of the form, $f(x) = 0$

The Corrector formula

$$x^c_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + f'(x_{n+1})} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Which is implicitly defined, in the sense that, we need to determine $[f'(x]_{n+1})$ to find x_{n+1} and hence we need a predictor formula for this. The predictor formula is the Euler's method

$$x^p_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

We discovered that while the Euler's method (2) has an error of order two; the Adeboye's predictor-Corrector method consisting of (1) and (2) has error of order four, if used once but converges to the solution in four of five iterative steps [24].

9.7 The Generalized Integration Formula for Polynomials of Degree n [25].

It is a well-known fact that there are integrals which cannot be found exactly. As a result of this, numerical integration methods are often used to approximate them after expressing the integrand in power series. Thus, there are numerical integration formulae of low degrees in composite forms which are repeatedly used to achieve convergence. It is observed that it will be more convenient, accurate and efficient if there is a general formula applicable to any degree of polynomial than the composite and rather cumbersome formula often used for approximating integrals of the form: $\int_a^b y(x)dx.$

So, we came up with a numerical formula of the form: $\int_a^b y(x)dx$

$= \sum_{i=0}^{2n} a_i\, y(x_c)$

Where, y(x) is a polynomial of degree, M = 2n.

Consequently, we arrived at the generalized formula:

$$A_{2n-1} = \begin{bmatrix} 111 & \cdots & 111 \cdots & 1 \\ -(n-1) & \cdots & 101 \cdots & n \\ (n-1)^2 & \cdots & 101 \cdots & I^2 \\ (n-1)^3 & \cdots & 101 \cdots & n^3 \\ \vdots & \vdots & \vdots \quad \vdots & \vdots \\ -(n-1)^{2n-1} & -(n-2)^{2n-1} & -101 \cdots & n^{(2n-1)} \end{bmatrix} \begin{bmatrix} a_2 \\ a_1 \\ \\ \vdots \\ \\ a_{2n-1} \end{bmatrix} = \begin{bmatrix} h \int_{-(n-1)}^{n} dx \\ h \int_{-(n-1)}^{n} x dx \\ \vdots \\ h \int_{-(n-1)}^{n} x^{2n-1} dx \end{bmatrix} \tag{3}$$

Which is a system of linear equations and which determines the constants? $a_0, a_1, \quad \cdots a_{2n}$, explicitly and hence the integral. It is interesting to note that all the integration rules ever developed are obtainable from this generalized formula. A very significant observation in this development is that the use of composite rules with a view to achieving convergence becomes superfluous or even obsolete. This is now widely used in numerical computation especially in Block Methods

9.8 Solution of Initial Value Problems and Boundary Value Problems by Numerical Methods [22].

This is the area of major interest and much was done and achieved as can be seen from the references. Of particular interest is the: H^2 - Galerkin method for the solution of two point boundary value problems of the form:

$L[y], f(x), x \varepsilon [a, b]$ and $y(a) = A;\ y(b) = B$.

The H^2 – Galerkin method is defined as $D^2\,(Lu, V) = D^2\,(f, V)$.

Where, $(Lu, V) = \int_a^b LuV\,dx,$ and $D^2 = \frac{d^2}{dx^2}$.

By integrating partially successively twice, we obtain the final form:

$(Lu, V^{iv}) = (f, V^{iv})$ which is exceptionally accurate when compared with the H^{1-} Galerkin method by Thomee and Wahlbin and the H^{-1} Galerkin by Rachford and Wheeler.

The H^2-Galerkin Method by Adeboye takes care of problems requiring C^1 and C^2 continuity (e.g. plates and slabs) which can not be effectively handled by any of the other methods [27-28].

Furthermore, H^2-Galerkin Method is superconvergent in the sense that, the error term is higher than expected. The most interesting thing about the Adeboye's method is that it is superconvergent for the functions and all its derivatives (see table 2 below)

9.9 Numerical Computations

Table 1: Solution of the Differential Equation

$y'' - 2(1 + 2x^2)y = 0; y(0) = 1, y(1) = e$

x	y_c	y_a	$\frac{dy}{dx}e$	$\frac{dy}{dx}a$	$\frac{dy^2}{dx^2}e$	$\frac{dy^2}{dx^2}a$
0	1	1	0	$3.9639E^{-05}$	2	2
1	1.01005	1.010054	0.20201	0.2020516	2.060502	2.060341
2	1.040811	1.040819	0.4163243	0.4163719	2.248151	2.248237
3	1.094174	1.094187	0.6565046	0.656564	2..582251	2.582406
4	1.173511	1.173528	0.9388087	0.9388884	3.098069	3.098329
5	1.284025	1.28405	1.284025	1.284139	3.952076	3,852497
6	1.433329	1.433365	1.719995	1.72016	4.930654	4.931268
7	1.632316	1.433365	1.719995	1.72016	4.930654	4.931268
8	1.896481	1.896558	3.03437	3.034603	8.647955	8.646956
9	2.247908	2.247997	4.046236	4.046062	11.77904	11.77006
10	2.718281	2.718281	5.436565	5.434301	16.3097	16.27087

Most numerical methods only deal with finding approximate solution for the function alone, because, it is a well-known fact that the approximation of function does not carry with it the approximation of its derivatives, but the H^2- Galerkin / fixed point iterative method of Adeboye does [27, 28, 29].

It is an obvious fact that a lot of work has been done for various types of problems in one –dimension. Often, the extension of these procedures to problems in two—dimensions is not immediate or easy. We succeeded in defining a C^0-collocation/finite element method for such problems, thus showing that some of the results obtained for problem in one – dimension can be duplicated in two dimensions for similar problems. The extensive use of the Hilbert Space and operators theories tremendously enhanced the achievement of superconvergence results for two dimensional problems.

9.10 **Reduced Integration**

It has been observed that the finite element superconvergence estimate is often obtained by using a technique called 'Reduced Integration' a rare phenomenon, which appears to violate all continuity requirements.

This closely associated with the use of very low order numerical quadratures on isoparametric elements by engineers.

Now, consider the equation

$$Lu^h = f$$

The weak form is given by

$(Lu^h, v) = (f, v)$

A formal integration by part is the bilinear form:

$[a(u]^h . v) = (f . v)$

which involve integrals of the form:

$$\iint p(x,y)\nabla\phi_i \quad \nabla\phi_j\,dx\,dy.$$

And which are actually integrated by numerical quadrature.

The relative difficulty in satisfying the continuity requirements especially in problems requiring C^1 or higher order continuity, made the engineers to consider ways of circumscribing or weakening the continuity requirement without at the same time, reducing the order of the accuracy of the method.

In Irons [30], it is stated that Gauss quadrature is just adequate and that it is the best numerical quadrature to be used. Also, it is observed that isoparametric elements are due to Gauss quadrature and optimal results are often obtained.

Zienkiewicz, et al [31] observed that in reduced integration, the resulting stiffness of displacement element is reduced since the order of numerical integration decreases (this is more or less the definition of reduced integration). Also, the convergence of the numerically integrated element is always quarranted, provided the integration order allows sufficient accuracy for an exact evaluation of the element volume.

The actual error obtained in practice compares favorably with the superconvergence results obtained analytically. The technique which often involves the use of least squres on isoparametric elements is called "reduced integration". Zienkiewicz and Hinton [124] termed it a case of apparently obtaining something for nothing.

It may be right therefore to define reduced integration as integration of a lower order than we would think, at first sight, strictly should be used. This could be put in another way as:

Minimum order of numerical quadrature = Optimum order of numerical quadrature.

Order of numerical Quadrature

The following questions are sufficient.

1. What is adequate order of numerical quadrature accuracy?
2. What is necessary for convergence?
3. What is necessary for a numerical quadrature to be the same as the finite element error?

The answers to these questions are found in the rule given by Zienkiewicz; et al [3] and which comes from Iron's intuition that the convergence of the finite element process will occur in elastic displacement problems if the integration is sufficient to integrate exactly the volume of the element.

In displacement based FEM, reduced integration is employed. This simply means that in the numerical integration of the element stiffness matrices the exact matrices are not evaluated. The method is simple to program and requires less computation time to establish the matrices, and with experience, acceptable results are frequently obtained. However, the technique can lead to very large errors. When reduced integration is used, some frequencies are better approximated than when using full integration, but among those few listed is a phantom frequency. Phantom frequencies do not physically exist but are only introduced by the reduced integration scheme. If however a dynamic step-by-step solution is performed, such phantom frequencies are not noticed.

Generally, standard lower-order quadrilateral isoparametric elements for solving elasticity problems have two serious drawbacks:

(i) They are overly stiff in bending problems.

(ii) They lock in incompressible problems (when the bulk modulus becomes infinite).

By blocking, we mean the inability of an element to perform accurately in an incompressible analysis regardless how refined the mesh is, due to over-constrained condition and insufficient active degrees of freedom. One remedy which has been used successfully is the reduced integration, though with a draw-back – spurious energy modes which lead to rank deficiency. In static cases, such spurious modes lead to singularity of the assembled stiffness matrix.

Reduced integration was first devised by Zienkiewicz, Taylor and Too in 1971[22, 23] to alleviate shear locking in plate bending. It saves a great deal of computational cost. For example, full integration of the stiffness of a 4-node quadrilateral element requires four integration points and for the 8-noded hexahedral element, eight integration points. Use of reduced integration only requires the evaluation of the matrices at one-point – the element's centroid! One-point quadrature thus provides tremendous cost benefits in linear and non-linear analysis. The reduced integration technique also provides the added benefit of eliminating spurious constraints in bending and incompressible applications as demonstrated by Makus and Hughes in 1978.

9.11 **Mathematical Modeling**

Using Mathematical Modeling we have been able to visualize many practical problems as mathematical problems and thus, able to give them logical and quantitative analysis and realistic solutions as against the verbose unscientific speculative and unrealistic qualitative analysis often employed by people, especially the uninitiated.

Some of the problems we considered here are:

1. The uncomfortable weather condition in Niger state
2. Optimizing the crop irrigation system
3. Oil exploration
4. The Nigeria Vision 20:2020; among others.

All these have been well researched and published in journals for the benefit of the society at large [34-44].

In the case of vision 20:2020, we observed that, since it was initiated, many scientists and economists have expressed doubt about its achievability, researchability or practicability because of some major infrastructural defects like unimaginably low power generation, poor transportation system, low level of industrialization, etc. Another major factor often mentioned is corruption and its debilitating effects on the economy.

However, we observed that power generation and corruption are very important determining factors and are bound to play decisive roles in the attainment or otherwise, of the vision. With the model, we were able to state the level of power generation and corruption index which can enhance the achievability of the vision.

While we viewed availability of power as positive and diffusive, we saw corruption as negative and reactionary. Then, we came up with a mathematical behavior with these two characteristics, the reaction-diffusion equation:

$$\frac{\partial E}{\partial t} = D\nabla^2 E + f(t, x, E)$$

where, E stands for the economy.

We came out with some very realistic and easily understandable results or predictions. Even people who have some obsession about or are averse to mathematics, were able to see this time around that mathematics is increasingly becoming a beautiful bride everyone would wish to embrace.

Specifically, we looked at the thirty leading economies in the world today, predicted where they could be in the year 2020 and use the model to predict the level of power generation on annual basis and the corruption index which could make Nigeria one of the 20 leading economies in the year 2020. When we presented this at the NMS conference at Unilorin, in the year 2009, one of the people advising the government on power generation was there and after the presentation, he came and confirmed that all our predictions are correct. He commended us for this and took a copy of the paper away with him. We also made it known that unemployment can be brought to the barest minimum and the economy will boom, if electricity supply is sufficient.

10.0 The Way Forward

There is no gainsaying the fact that the best numerical method is the iterative one because, any good iterative method is convergent and a convergent iterative method is self-correcting and hence very easy for the computer to process. Therefore, having found a very accurate and efficient iterative method for the solution of ordinary differential equations, one is inspired to explore the possibility of extending it to the solution of partial differential equations. However, this is not always that easy to achieve, but according to Vergil "*Possum quia posse Videntur*". That is "to resolve upon attainment is often attainment itself".

Therefore, having resolved to extend the hybrid methods of Galerkin – fixed point iteration procedures to the solution of partial differential equations, the attainability is just a matter of time. Also, it is a good research area, especially for those with inventive and lucid minds.

11.0 General Observation Conclusion and Recommendations

In 1961 when the Russians sent a man to space, the Americans became jittery because of the military opportunities therein. The American President, J. F Kennedy, immediately summoned the experts to find out why USA was lagging behind in space exploration. The experts conducted a research into this and came out with their finding – that it was due to lack of sufficient knowledge of mathematics on the part of American students. Then, immediately, steps were taken to ameliorate this deficiency and by 1969, 8 years later, Americans landed on the moon!

Thus we have seen how mathematics has come to be regarded as a language –a means of communication and description which is increasingly being used in all areas of human endeavour by those who are able to study it. Mathematics is needed for the development, maintenance, understanding, quantification and record keeping of our society. Since no society is static and the desire for higher heights in science and technology will continue to increase the demand on mathematics will ever be on the increase. In view of the universal importance of mathematics to man on earth, it becomes compulsory that those charged with education should find ways of involving more of the younger generation of our days in the study of mathematics. Any nation that cannot get school children involved and be interested in mathematics, will never attain true social , economic ,scientific and technological independence. Such a nation will continue to look up to those other nations of the world which through sound mathematics education have become world powers, with sound economic, scientific and technological bases, for her needs, even in matters of political guidance.

The educational system in Nigeria as at now is in a state of comma as a result of neglect. The rule then was "acquire now, neglect or even abandon later". The governments acquired all the educational institutions from primary to tertiary level in the late 1970's only to neglect or even abandon them in the early 1980's. Some states established universities only to boost their egos. They never considered the cost of running a university before establishing one. They would want to hire lecturers at the rate of two for a kobo! I wish to advise the NUC to add to the conditions to be satisfied before establishing a university that the minimum conditions of service acceptable are those obtainable at the federal universities and this will put paid to the unbridled histrionics often engaged in by the owners. For this, Niger state government deserves commendation because the conditions of service at the IBB University, Lapai are at par with those obtainable at the federal universities. In fact, it is the only state which established a university and imposed the conditions on itself from the start.

This is one of the ways to ensure good standard and quality in the university system. In the first republic, a professor was earning more than a minister or any legislator; and in the USA, the annual salary of the president of the United States is \$216,000 as at march, 2013, whereas there are professors earning as much as \$250,000. In the state of New York, USA, the salary of the governor is less than that of the primary school headmaster/headmistress.

In August 1939, Albert Einstein in a letter to the president of the USA, Franklin D. Roosevelt stated that "In the course of the last four months it has been made almost certain that it may become possible to set up a nuclear chain reaction in a large mass of uranium by which vast amounts of power and large quantities of new radium-like elements would be generated. The new phenomenon would lead also to the construction of bombs".

The American science at that time was not involved in any significant dialogue with the society. Private industry which had yet to discover that pure science pays off, was so uninterested in the university scientists that to land jobs in it, they often had to conceal their academic achievements. The federal government's outlay for scientific research and development was a paltry sum of about fifty million dollars most of which was devoted to agriculture. "The Scientist was pictured as a queer sort of chap who lived a hermit-like existence, dedicated to goals not much different from those of the alchemist of medieval times" (John Brooks, [65]. The public stereotype of the scientist enshrined in the movies and in popular fiction held him to be "mad, unpractical but visionary, eccentric but harmless". Einstein's simple formula: $E = \frac{1}{2}mc^2$ was seen to be too simple to be important and therefore could not have any practical application.

In 1940, the wheels of a federal atomic bomb project began turning. In June, 1942. the Manhattan project, as it was named, was carried out by a group made up of Physicists, Chemists and Mathematicians (Newsweek, July 29, 1985). After the dropping of the bomb euphemistically coded 'Little boy' which pulverized Hiroshima on August 6 1945, "the esoteric scribbling of the amiable eccentrics were shown to have contained the secret relationship between matter and energy and the professors were shown to be most practical men alive!" (John Brooks, 1982). Suddenly, the harmless madmen were revealed as sorcerers, holding such destructive power as no man had ever held before! There and then "the dialogue between science and American society was to become the principal dialogue of the age".

The annual federal outlay for scientific research and development skyrocketed to about twenty five billion dollars. Ever since, USA Congress' instinctive reaction to a request for funds for science is to raise rather than lower it, and the magic word "research" has become "the master key that unlocks the Treasury to him who utters it". Thus the American society has learnt that what looks impractical in science may not be. Here unfortunately, Nigerian government and indeed the Nigerian society are yet to learn that the progress of science and technology is obviously the main spring of transformation, and of course, the transformation agenda of the present government.

In 1961, Dwight D. Eisenhower, the then President of the United States of America remarked in his farewell speech that: The free university, historically the fountainhead of free ideas and scientific discoveries, has experienced a revolution in the conduct of research, partly because of the huge costs involved, a government contract virtually a substitute for intellectual curiosity. The prospect of domination of the nation's scholars by federal appointment, project allocation and the power of money is ever present and is gravely to be regarded. This exactly, is the situation in Nigeria today and in the words of Eisenhower 'it is gravely to be regarded'.

It may be pertinent to add that there is a direct national budget for mathematical research in the USA. In a recent analysis developed for the Wall Street Journal which evaluated 200 professions including politics, the mathematician has the top job in the USA, placed first in terms of income, good environment, employment outlook, physical demands and low stress. Actuarian and statistician ranked second and third respectively and these are also are also mathematicians. In Nigeria here today, science and mathematics are no longer perceived as offering desirable career opportunities. Brilliant mathematics students who could have chosen careers in mathematics now go for Medicine, Engineering and even Law! In the 1960's and 1970's the best students in mathematics and physics at Higher School Certificate (H. Sc). Math/Math/Physics) were taking mathematics as their first choice in the universities and all other students envied them.

It is very clear to everyone now that the government alone cannot be responsible for education. But there is something the government can do all on its own and that is supervision. If the government is spending so much on education, why is it not making sure that it gets good value for its money? The schools need supervision. In fact it is obvious that the government in Nigeria is always shy of supervision. Another example is the environmental problem: why for God's sake should the government not set up a sanitary inspectorate division, instead of the cosmetic compulsory sanitation day which is a partial solution? Why are governments afraid of supervising the various systems? In mathematics, we always look for complete solutions to problems we do not believe in partial solutions. Looking for a partial solution is like begging the issue or pecking the periphery but this is what the government does in Nigeria all the time! Partial solutions are dangerous, egregious folly, time-wasting and can be very costly, if not costlier than the complete solution.

11.1 Ranking of Universities

Every time anybody talks about the "falling standard of education" in Nigeria, the offending Adam or the villain of the piece is the university system. Everybody pooh-poohs, lampoons or even vilifies the universities for bad showing or not being listed among the first one hundred or first five hundred or so universities in the world! I find this, to say the least, very ridiculous and to me it portrays such people as uninformed. Essentially they do not understand the proper use of numbers or statistics. If people understand the proper use of statistics or numbers, they would not be talking of the listing of world universities according to positions especially when there are about 50,000 universities. Perhaps it may be pertinent to state here that as at 2009, India had over 8000 universities and USA had over 5000 universities. Even Bangladesh, a third world country had over 1200 universities. Indonesia had over 1200 too.

Judging by these numbers alone I would wish to submit that it is not quite proper to assess universities by listing. Statistically or mathematically, the best ways of ranking these large numbers of world universities are:

i. By percentage grouping; that is the first top one percent or five percent, ten percent, etc in the world
ii. By disciplines viz: one of the best in Medicine, Computer science, Applied mathematics, etc in the world.

Even post graduate schools the world over often ask candidates' referees to state whether a candidate is among the top 5%, 10% of his peers and in most cases the peers are in tens. Why should we look at positions of universities differently? *Verbum sat sapient* (a word is sufficient for the wise).

11.2 University of Technology

A university is often referred to as an Ivory Tower and by definition; an ivory tower is a place of seclusion or retreat from the realities of life. So one may wish to ask; is the university just an ivory tower? Is it involved in any serious dialogue with the society or the immediate environment? A university of technology is supposed to train professionals who will be job providers and not job seekers, undertake applied research development in science and technology and service innovations by combining learning and research and development (that is learning by development) while conventional universities provide scientific and artistic education. *Res ipsa liquitur* (the thing speaks for itself).

There is this Greek notion of self-reliance "I have boot-strapped my way; everyone has got to boot-strap his/her way". That is to say, we should not expect USA or Britain to come and develop Nigeria for us. They always put their money where their interests are! Many of our problems are as a result of neglect due to negligence or failure to boot-strap. Or else, how does one explain the epileptic power supply when we have gas and oil in abundance; the congestion and carnage on our roads with the trailers as kings of the road while the railway system is almost non-existent? I am sure many of us are aware that the fastest and safest land transportation system is the railways. The Chinese now have a train that does 560km/hr! Well, like Shakespeare wrote in Henry IV, and I quote "I spy life peering but, I dare not say how near the tidings of our comfort are". Perhaps, one may take solace in this Latin saying: *"Dimidium facti qui coepit habet"* (that is, half is done when the beginning is done).

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